Univariate Calculus Applications with the HP 49 G Calculator

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Univariate calculus applications

In this document we present some examples of applications of derivatives and integrals of one variable in selected physical and engineering sciences.

Dynamics: rectilinear motion applications

Let s(t) represent the position along a straight-line path of a particle as a function of time t. By definition the velocity of the particle is v(t) = ds/dt, and its acceleration is a(t) = dv/dt = d^2s/dt^2. Another relationship that is commonly used results from eliminating dt from the equations for v(t) and a(t), which results in dv/a = ds/v or v·dv = a·ds. The latter result is useful when you are given a = a(s).

Example 1 – Given *s(t) = t*-sin *t*, plot the displacement, velocity, and acceleration of the particle as a function of t in the interval [0, 5].

- First, enter the expression for s(t):

  [EQW] [ALPHA][↰][S] [↰][( )] [ALPHA][↰][T] [▶] [↱][=] [ALPHA][↰][T] [▶] [-] [SIN][ALPHA][↰][T] [ENTER] [↰][DEF]

  This creates variable [s].

- Next, calculate the velocity v(t):

  [EQW] [ALPHA][↰][V] [↰][( )] [ALPHA][↰][T] [▶] [↱][=] [↱][∂][ALPHA][↰][T] [▶][ALPHA][↰][S] [↰][( )] [ALPHA][↰][T] [ENTER] [↱][EVAL] [↰][DEF]

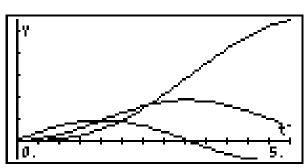
  This creates variable [v].

- The next step is to calculate the acceleration a(t):

  [EQW] [ALPHA][↰][A] [↰][( )] [ALPHA][↰][T] [▶] [↱][=] [↱][∂][ALPHA][↰][T] [▶][ALPHA][↰][V] [↰][( )] [ALPHA][↰][T] [ENTER] [↱][EVAL] [↰][DEF]

  This creates variable [a].

- To plot these functions you need to load the list {'s(t)' 'v(t)' 'a(t)'} into EQ, change the independent variable to t, change the range of values of t from 0 to 5, use AUTO to generate the range of values for the y-axis, and proceed to create the plot. For details in creating FUNCTION type plots see the examples in Chapter 11. The result is the following plot:



Example 2 – Given the speed of a particle as function of time t, v(t) = exp(-t/5), obtain an expression for the position of the particle s(t) if the particle started at s = -5 when t = 2. Also, find the acceleration of the particle at t = 1.

- First, we define the velocity function:

[EQW] [ALPHA][↰][V] [↰][( )] [ALPHA][↰][T] [▶] [↱][=] [↰][e^x] [+/-] [ALPHA][↰][T] [÷][5] [ENTER] [↰][DEF]

This creates the variable [v   ]

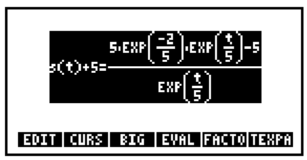
- From the definition of velocity, v(t) = ds/dt, we can write ds = v(t)dt, and integrate

$$\int_{-5}^{s} ds = \int_{2}^{t} v(t)dt$$

by using:

[EQW] [↱][∫] [5][+/-][▶] [ALPHA][↰][S] [↰][( )] [ALPHA][↰][T][▶] [1][▶] [ALPHA][↰][S][▶] [↱][=] [↱][∫] [2][▶] [ALPHA][↰][T][▶] [ALPHA][↰][V] [↰][( )] [ALPHA][↰][T][▶] [ALPHA][↰][T] [▲][▲][▲] [EVAL]

This is the result shown in the equation writer screen (small font):



- Press [ENTER] [↱][‘][5] [↱][=][5][ENTER][–] [↱][ALG][EXPAN] to eliminate the 5 from the left-hand side of the equation (using ISOL does not work here).   The result is:

‘s(t)=((5*EXP(-2/5)-5)*EXP(t/5)-5)/EXP(t/5)’.

- To find the acceleration of the particle can be found by using:

[EQW] [ALPHA][↰][A] [↰][( )] [ALPHA][↰][T] [▶] [↱][=] [↱][∂][ALPHA][↰][T] [▶][ALPHA][↰][V] [↰][( )] [ALPHA][↰][T] [ENTER] [↱][EVAL] [↰][DEF]

The result is ‘a(t)=-1/(5*EXP(t/5))’.

- To find the value of the acceleration at t = 1, use:

[↱][‘][ALPHA][↰][T] [↱][=] [1] [ENTER] [↱][ALG][SUBST]

The result is: ‘a(1) = -1/(5*EXP(1/5))’.

- To get a numerical value out of this expression, we need to separate the expression as follows:

[↰][PRG][TYPE][OBJ→][⇦][⇦] [↱][→NUM]

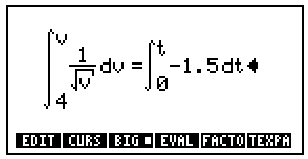
The result is –0.163746150616.

Example 3 - A particle is moving with an acceleration $a = -1.5\ v^{1/2}$, with $v = 4$, when $t = 0$. Determine and expression for the velocity, *v(t)*, and evaluate the velocity at $t = 2$.

- From the definition of acceleration, $a = dv/dt$, and using the initial conditions indicated above, we can write the following integral equation:

$$\int_4^v \frac{dv}{\sqrt{v}} = -1.5\int_0^t dt$$

- Enter the integral in a similar fashion as done in Example 2. The equation should look like this in the equation writer:



- Make sure your CAS mode is set to `Exact (-105 CF)`. Press [ENTER][↱][EVAL] to calculate the integrals in the equation. When asked for `Approx` mode on, choose YES, and press [OK]. The result is '-4.+2.* √v=-(1.5*t)'

- Then, use [↱]['][ALPHA][↰][V] [↰][S.SLV][ISOL] to obtain: 'v=.5625*t^2+ –3*t+4.'

- To evaluate this expression at t = 2, use:

[↱]['] [ALPHA][↰][T] [↱][=] [2][ENTER] [↱][ALG][SUBST] [↱][EVAL]

The result is 'v=0.25'.

Dynamics: motion in polar coordinates

Example 1 – Finding velocity and acceleration in the radial direction given r = f(θ(t)).

When describing the trajectory of a particle in polar coordinates, r = f(θ), we are usually required to find the derivatives,

$$v_r = r' = dr/dt, \text{ and } a_r = r'' = dv_r/dt = d^2r/dt^2.$$

If θ(t) is given, then, we can just replace it into f(θ), to get r = g(t) = f(θ(t)). For example, if r = 2.5 sinθ, and, $\theta = 3.5t^2 - 2t$, we can simply write $r = 2.5 \sin(3.5t^2 - 2t)$.

To obtain the derivatives using the HP48G or GX calculator, we enter the expression for r(t) in the display and use the [↱][∂] keystroke sequence. For example, enter the expression,

```
'2.5*SIN(3.5*t^2 - 2*t)'
```

in stack level 1 of the calculator, and store it into the variable r by using

[↱]['][ALPHA][↰][R][STO▶].

Then, calculate the derivative by using:

[VAR][r] [↦]['] [ALPHA][↤][T][ENTER] [↦][∂]

Simplify the expression by using the function COLCT (COLleCT), available through the command catalog: [CAT][ALPHA][C] (find COLCT)[OK], to get an expression for r'(t) as

```
'(17.5*t-5.)*COS(3.5*t^2 - 2*t)'
```

Save this expression into variable *rt*, by using:

[↦]['][ALPHA][↤][R] [ALPHA][↤][T] [STO▶].

To obtain the second derivative of r with respect to t, use:

[VAR][rt] [↦]['] [ALPHA][↤][T][ENTER] [↦][∂][CAT][OK]
(Note: the function COLCT should be readily available)

The result is:

```
'-((122.5*t^2.+-70.*t+10.)*SIN(3.5*t^2.-2.*t)-17.5*COS(3.5*t^2-2.*t))'
```

or,

$$r'' = -((122.5t^2-70t+10)\sin(3.5t^2-2t)-17.5\cos(3.5t^2-2t)).$$

Example 2 -- Finding velocity and acceleration in the radial direction given r = f(θ).

If you want to find the derivatives r' and r" for r = f(θ), where θ is not given explicitly as a function of time, you can still use the HP 49 G calculator to obtain expressions in terms of $\theta' = d\theta/dt$, and $\theta'' = d^2\theta/dt^2$. We will need to write the expression for r as f(θ(t)) and take derivatives with respect to t.

For example, given r = 2.5 sinθ, evaluate r' and r" when θ = 0.5 rad, θ' = -3.5 rad/s, and $\theta'' = 2\ rad/s^2$.
We will write in the calculator the following expression:

'2.5*SIN(θ(t))'

and save it into r:

[VAR][↤][r].

Then, calculate the derivative dr/dt by using:

[VAR][r] [↦]['] [ALPHA][↤][T][ENTER] [↦][∂]

We get the result:

'2.5*(COS(θ(t)))* d1θ(t))',

where *d1θ(t)* represents θ' = dθ/dt. In other words, our result is

$$r' = 2.5 \cdot \cos\theta \cdot \theta'.$$

Save the expression in stack level 1 into variable *rt*, by using:

[VAR][↤][rt].

To obtain the second derivative of r with respect to t, use:

[VAR][rt] [↱]['] [ALPHA][↰][T][ENTER] [↱][∂][CAT][OK]

(Note: the function COLCT should be readily available)

The resulting expression is '2.5*COS(θ(t))*d1d1θ(t)-2.5*d1θ(t)^2.*sin(θ(t))'.

With the understanding that d1d1θ represents the second derivative of θ with respect to t, i.e., θ'', we can write:

$$r'' = -2.5\cdot [\theta']^2 \cdot \sin\theta + 2.5\cdot\cos\theta \cdot \theta''.$$

At this point we can replace the values given earlier for θ'and θ". Keeping the last expression in stack level 1 create the following list:

```
{ ‘θ(t)=.5’ ‘d1θ(t)=-3.5’ ‘d1d1θ(t)=2’} [ENTER]
```

and use the keystroke sequence:

[↰][DEF]

to define the three "functions" (actually constant values) to be able to evaluate the expressions for the derivatives. Now, enter

[↱][EVAL]

to get the value -10.2944943103.

To evaluate the first derivative r'(t) and the position r(t) use:

[VAR] [rt] [↱][EVAL] Result: -7.67884741655

[r] [↱][EVAL] Result: 1.19856384651

When done, you may want to purge all the variables defined here by creating the list:

```
{‘θ’ ‘d1θ’ ‘d1d1θ’ ‘rt’ ‘r’} [ENTER]
```

and using

[TOOL][PURGE].

The function COLCT

The function COLCT belongs in the old HP 48 G/G+/GX SYMBOLIC menu, but it is still available in the HP 49 G calculator through the command catalog as shown in the examples above.

Probability: Calculations with continuous random variables

For probability distributions of continuous random variables, probabilities are calculated using the *cumulative distribution function* (*CDF*), F(x) = P(X ≤x). The definition of the CDF for continuous variables utilizes definite integrals. We can use the HP48G series calculator to evaluate such integrals either symbolically or numerically. Following we present some examples within a new subdirectory HOME\STATS\INTS :

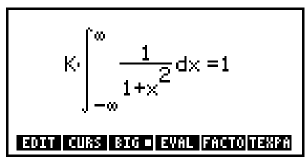
1) Suppose that the pdf of a continuous random variable is given by f(x) = K/(1+x^2), for - ∞ < x < ∞. We are asked to find the value of K. By definition,

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

for this particular case, we can write

$$K \cdot \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = 1.$$

It should be straightforward to type this equation in the equation writer to produce:



Press [ENTER][↱][EVAL], to get the result: ‘K*π=1’. Of course, you can easily figure out that K = 1/π, and find this value by using: [↰][π][1/x][↱][→NUM], i.e., K = 0.318309886184.

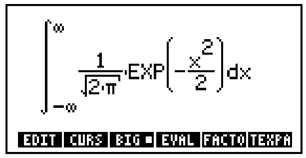
2) Consider the expression for the Standardized Normal distribution,

$$f(x) = \frac{1}{\sqrt{2 \cdot \pi}} \exp(-\frac{x^2}{2}).$$

Prove that, for this distribution, $_{-\infty}\int^{+\infty}$ *f(x)dx=1*. First, type in the expression

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2 \cdot \pi}} \exp(-\frac{x^2}{2})dx$$

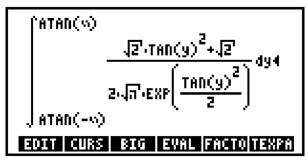
in the equation writer, to produce:



A direct evaluation, by using [ENTER][↦][EVAL] produces no numerical result. One possibility is to use the substitution $x = \tan y$. Thus, having the integral listed in stack level 1, enter:

‘x=TAN(y)’ [ENTER][↦][ALG][SUBST].

Next, press [▼] to activate the equation writer, and enter [▼][▶][▶][▶][⇦]. Your equation writer screen should now look like the figure below:



Let’s evaluate the limits of integration by using: [▶][▼][EVAL] [▶][EVAL]. The limits of integration now become –π/2 and π /2. Press [ENTER] to exit the equation writer, change CAS mode to `Approx`, by using [1][0][5][+/-][ALPHA][ALPHA][S][F][ENTER], and use [↦][→NUM] to obtain a numerical value. Be aware that it takes the calculator up to five minutes to obtain the numerical result: 0.999999999996, which is as close to 1.0 as we can get.

Note: The integral calculated above is an improper integral (i.e., one or both limits are ±∞). You can use the transformation $x = \tan(y)$ to convert the improper integral into a proper integral . The transformation is expressed by the following formula:

$$\int_a^b f(x)dx = \int_{\arctan(a)}^{\arctan(b)} f(\tan y)(1+\tan^2 y)dy$$

If $a = -\infty$, then $\arctan(a) = -\pi/2$. Also, if $b = \infty$, then $\arctan(b) = \pi/2$. Also, $\arctan(0) = 0$.

Note: Some integrals, for example,

$$\int_1^\infty \frac{dx}{x} = \int_{\pi/4}^{\pi/2} \frac{(1+\tan^2 y)}{\tan y} dy$$

do not converge to a value. And, in most cases, there is no way to tell from just looking at the integral that such is the case. (The case above is simple, since we know that $_1\int^x dx/x = \ln(x)$, therefore, $_1\int^\infty dx/x = \ln(\infty) = \infty$.)

---

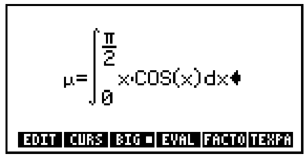
3) On to a simpler example: If $f(x) = \cos x$, for $0 < x < \pi/2$, and $f(x) = 0$, *elsewhere*, find $P(0<X<\pi/4)$. We need to calculate the following integral:

$$P(0 < X < \frac{\pi}{4}) = \int_0^{\pi/4} \cos(x)dx$$

Use the following:

[↱][‘] [↱][∫] [0] [↱][,] [↰][π][÷][4] [↱][,] [COS] [X] [▶] [↱][,] [X] [ENTER] [↱][EVAL].
The result is ‘√2/2’. If you use [↱][→NUM], this result is shown as 0.707106781185.

4) To calculate the mean [$\mu = \int x \cdot f(x)dx$] of the pdf in case 3, enter the following integral:

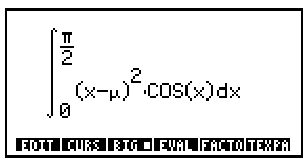


Press [▲][▲][EVAL] to get the result μ = (π-2)/2.

Store this value in a variable called μ by using:

[ENTER][↰][PRG][TYPE][OBJ→][⇦][⇦] [▶] [STO▶]

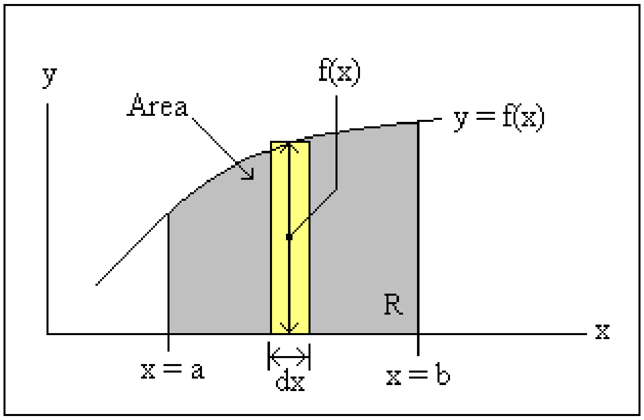
5) To calculate the variance [$\sigma^2 = \int (x-\mu)^2 f(x)dx$] of the pdf defined in 2, whose mean was calculated in 4, type in the following integral:



The display now shows: 0.141592….(i.e., σ^2 = 0.141592…).

Statics: properties of areas

Consider the region R in the x-y plane limited by the x-axis (y = 0), the curve y = f(x), and the vertical lines x = a and x = b as sketched in the figure below.



The small rectangle of width dx and height f(x) is a representative *differential of area* dA = f(x) dx for the region of interest. The area of the region will be calculated by adding the infinitesimal area elements between the values of x = a and x = b, i.e., by calculating the integral

$$A = \int_R dA = \int_a^b f(x)dx$$

The product x·dA = dM_y is the infinitesimal first moment of the differential of area dA with respect to the y-axis. Here, x represents the location of the *centroid* (center of mass, center of gravity) of the infinitesimal rectangle dA. Integrating dM_y over the values of x =a and x = b, we obtain the first moment of the area with respect to the y-axis, i.e.,

$$M_y = \int_R dM_y = \int_a^b x \cdot f(x)dx$$

Using the element of area shown above, it is possible to define a differential first moment of dA with respect to the x-axis as dM_x = (y/2)·dA, since y/2 represents the location of the centroid of dA with respect to the x-axis. Thus, the first moment of the region R with respect to the x-axis is given by

$$M_x = \int_R dM_x = \frac{1}{2} \cdot \int_a^b [f(x)]^2 dx$$

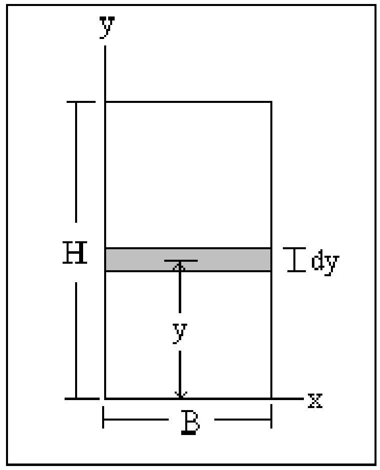
The first moments of the area, My and Mx, are used to calculate the coordinates of the centroid,

$\bar{x}$ = My/A, and $\bar{y}$ = Mx/A.

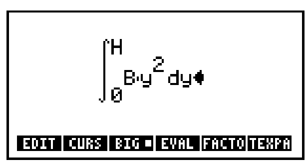
The quantity $dI_y = x^2 \cdot dA$ is referred to as the moment of inertia of the infinitesimal area dA with respect to the y-axis. The moment of inertia of the region R with respect to the y-axis is given by

$$I_y = \int_R dI_y = \int_a^b x^2 \cdot f(x)dx$$

The moment of inertia of the differential of area dA with respect to the x-axis, dI_x, is not as simple to write as that with respect to the y-axis, dI_y. The expression for dIx follows from the expression for the moment of inertia of a rectangle. Consider the rectangle shown in the figure below.



If we use the horizontal strip of the rectangle as a differential of area, dA_H = B dy, we can write $(dI_x)_R$ = y^2dA^H = By^2dy, where the sub-index R stands for "rectangle." The moment of inertia of the rectangle with respect to the x axis can be calculated as the integral $\int_R By^2dy$, between y = 0 and y = H. Using the HP 49 G calculator, the integral should look like this:



Press [ENTER][↱][EVAL] to get the result $(I_x)_R = B\cdot H^3/3$. Thus, the moment of inertia of dA, the vertical infinitesimal rectangle of width dx and height f(x), is given by $dI_x = (1/3)(dx)[f(x)]^3 = (1/3)[f(x)]^3dx$, and the moment of inertia of the region R with respect to the x-axis is calculated as

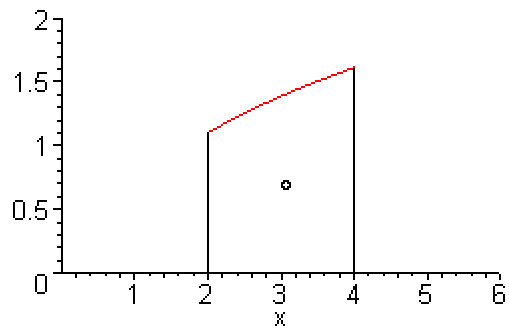
$$I_x = \int_R dI_x = \frac{1}{3}\int_a^b [f(x)]^3 dx.$$

The moment of inertia of the region R with respect to the origin (or, more properly, with respect to the z-axis, perpendicular to the x-y plane) is defined as

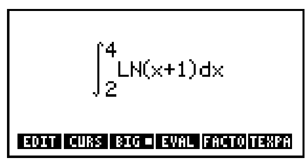
$$I_o = I_x + I_y = \int_a^b x^2 f(x)\cdot dx + \frac{1}{3}\int_a^b [f(x)]^3 dx.$$

Associated with the concept of moment of inertia is the idea of a radius of gyration. The radius of gyration about the y-axis is given by $k_y = (I_y/A)^{1/2}$, the radius of gyration about the x-axis is $k_x = (I_x/A)^{1/2}$, and that about the origin (or z-axis) is $k_o = (I_o/A)^{1/2}$.

Example: The figure below shows the region R defined by $0 < y < \ln(x+1)$, $2<x<4$. Use your HP 49 G calculator to obtain the area of the region , the coordinates of the centroid, the moments of inertia and radii of gyration about the x- and y-axis, and about the origin.

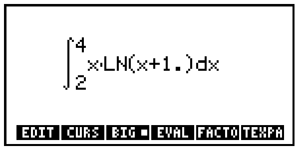


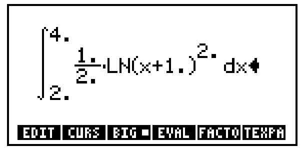
The solution requires you to use a = 2, b = 4, and f(x) = ln(x+1). Thus, the area would be calculated by

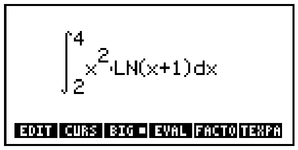


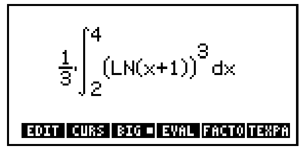
which produces the value ‘ -(3*LN(3)-(5*LN(5)-2))’, or A = 2.75135269614. Store this value in a variable called A.

The following screens shown the integrals corresponding to the first moments My, Mx, and the moments of inertia Iy, and Ix:





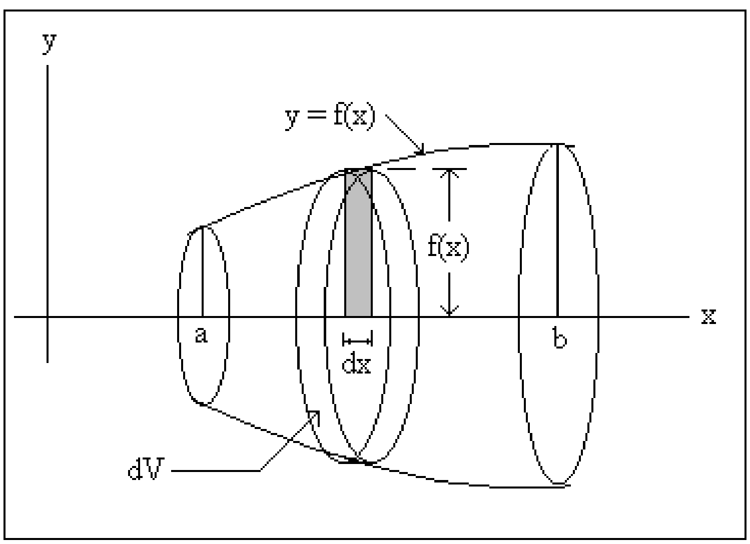




Once evaluated, the following results are obtained: My = 8.42286591025, Mx = 1.91394984757, Iy = 26.6864290145, and Ix = 1.7943172852. From these values we get, $\bar{x}$ = My/A = 3.06135448282, $\bar{y}$ = Mx/A = 0.695639584934, $k_y = (I_y/A)^{1/2}$ = 3.11438356332, and $k_x = (I_x/A)^{1/2}$ = 0.807563138712. Also, Io = Ix + Iy = 28.4807462997, and $k_o = (I_o/A)^{1/2}$ = 3.21738142011.

Dynamics: properties of solids of revolution – disk method

Consider the solid of revolution resulting from the rotation of the region R = {0 < y < f(x), a<x<b} about the x-axis, as illustrated in the figure below.



As the region rotates about the x-axis, the element of area – the shaded rectangle of width dx and height f(x) – generates a cylinder, or disk, of radius f(x) and height dx. The volume of this elementary cylinder (differential of volume) is

$$dV = \pi \cdot [f(x)]^2 \cdot dx.$$

The volume of the entire solid of revolution, contained in a<x<b, is, therefore,

$$V = \int_R dV = \int_a^b \pi \cdot [f(x)]^2 \cdot dx.$$

The external area of this elementary cylinder (differential of external area) is

$$dA_s = 2 \cdot \pi \cdot f(x) \cdot dx.$$

The total external area of the solid of revolution is, therefore,

$$A_s = \int_R dA_s = \int_a^b 2 \cdot \pi \cdot f(x) \cdot dx.$$

Let ρ represent the density (mass/volume) of the material composing the solid of revolution. By definition, $\rho = dm/dV$, where m represents mass. Therefore, the mass of the element of volume (differential of mass) is

$$dm = \rho \cdot dV = \pi \cdot \rho \cdot [f(x)]^2 \cdot dx.$$

This expression for dm applies if ρ is a constant. We can let the density vary with x, r(x), in which case the differential of mass is given by

$$dm = \rho \cdot dV = \pi \cdot \rho(x) \cdot [f(x)]^2 \cdot dx.$$

For constant ρ, the mass of the solid of revolution is simply

$$m = \rho \cdot V,$$

with V as calculated earlier.

For $\rho = \rho(x)$, the mass of the solid of revolution is to be calculated with the integral

$$m = \int_a^b \pi \cdot \rho(x) \cdot [f(x)]^2 \cdot dx.$$

The x-axis is an axis of symmetry for the solid of revolution, therefore, the y-coordinate of its center of mass is $\bar{y} = 0$. Assuming, in general, that $\rho = \rho(x)$, the first moment of the differential of mass with respect to the y-axis is given by

$$dM_y = x \cdot dm = x \cdot \rho \cdot dV = \pi \cdot x \cdot \rho(x) \cdot [f(x)]^2 \cdot dx.$$

The first moment of the solid of revolution with respect to the y-axis is, therefore,

$$M_y = \int_a^b \pi \cdot x \cdot \rho(x) \cdot [f(x)]^2 \cdot dx.$$

The x-coordinate of the solid body's center of mass is given by

$$\bar{x} = M_y/m.$$

The moment of inertia of the differential of mass with respect to the y-axis is given by

$$dI_y = x^2 \cdot dm = x^2 \cdot \rho \cdot dV = \pi \cdot x^2 \cdot \rho(x) \cdot [f(x)]^2 \cdot dx.$$

This expression results from the conditions of symmetry of the differential of mass about the y-axis that allows us to consider the inertial effect of the mass differential as that of a particle of mass dm located at a distance x from the y-axis. The moment of inertia with respect to the y-axis is calculated using the following integral

$$I_y = \int_R dI_y = \int_R x^2 \cdot dm = \int_R x^2 \cdot \rho \cdot dV = \int_a^b \pi \cdot x^2 \cdot \rho(x) \cdot [f(x)]^2 \cdot dx.$$

To calculate the moment of inertia of the cylindrical differential of mass from the solid of revolution we need to use the expression for the moment of inertia of a cylinder of radius R and height H about its axis. This result, which is proved later in the book by using double integrals in polar coordinates, is given by

$$(I_y)_R = \frac{1}{2} \cdot \pi \cdot \rho \cdot H \cdot R^4.$$

Using this result with the elementary disk in the solid of revolution provides an expression for the differential of moment of inertia with respect to the x-axis:

$$dI_y = \frac{1}{2} \cdot \pi \cdot \rho(x) \cdot [f(x)]^4 \cdot dx.$$

Thus, the moment of inertia of the solid of revolution with respect to the x-axis will be given by the integral

$$I_x = \int_a^b \frac{1}{2} \cdot \pi \cdot \rho(x) \cdot [f(x)]^4 \cdot dx.$$

Radii of gyration of the solid of revolution with respect to the y- and x-axes, respectively, are given by

$k_y = (I_y/m)^{1/2}$, and $k_x = (I_x/m)^{1/2}$.

__Example__ - Consider the region R = { 0 < y < ln(x+1), 2<x<4} shown in the figure below. The region rotates about the x-axis generating the solid of revolution sketched in the figure below. Assuming that the density of the solid is given by ρ(x) = exp(-x/4), calculate the solid's volume, exterior area, mass, x-coordinate of its center of mass, moments of inertia and radii of gyration with respect to the y- and x-axes.

The approach I suggest for calculating the required properties for the solid of revolution is to define the functions f(x) = ln(x+1) and ρ(x) = exp(-x/4) in the HP 49 G calculator. You can also store the values of a=2 and b=4 in the calculator, and then simply type in the formulas shown earlier to obtain the different properties. Here is how to define the functions:

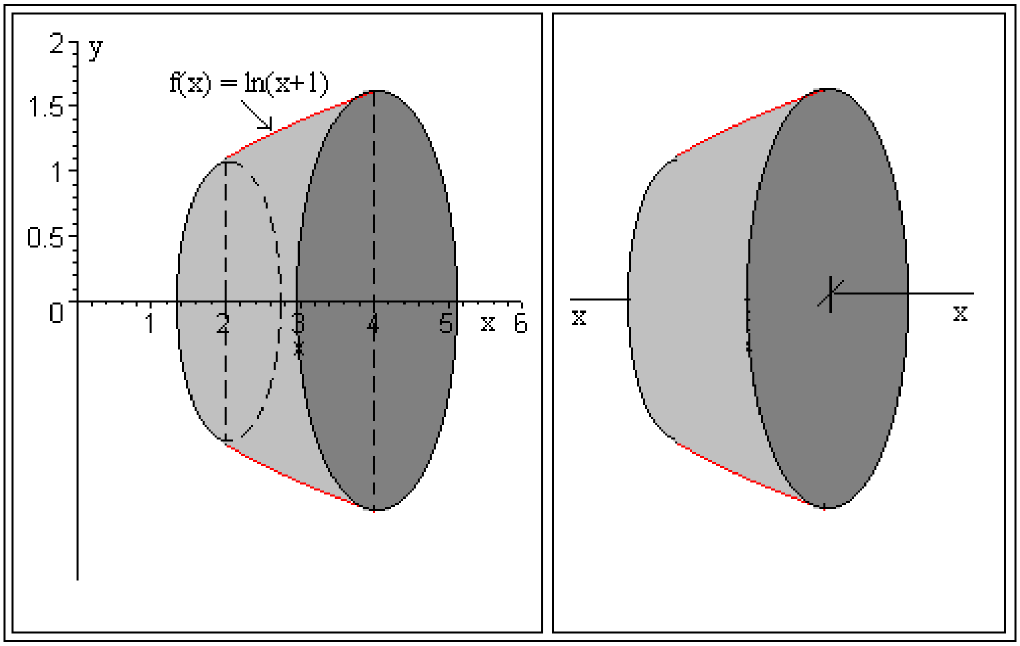
[EQW] [ALPHA][↰][F] [↰][( )] [ALPHA][↰][X] [▶] [↱][=] [↱][LN] [ALPHA][↰][X] [+][1]
[ENTER] [↰][DEF]

This operation creates the variable [f].

To create the density function use:

[EQW] [↱][CHARS] (find the character ρ) [ECHO1] [↰][( )] [ALPHA][↰][X] [▶] [↱][=] [↰][e^x]
[ALPHA][↰][X] [+/-] [÷][4] [ENTER] [↰][DEF]

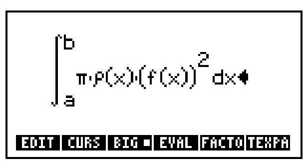
This operation creates the variable [ρ].



The next step is to store the values of a and b:

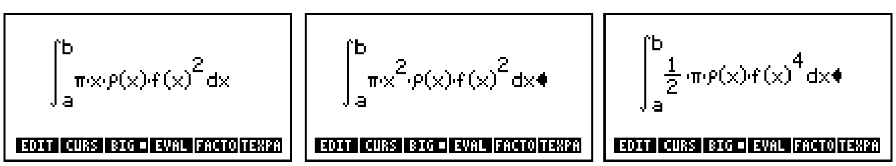
[2] [↱][‘][ALPHA][↰][A][STO▶]   [4] [↱][‘][ALPHA][↰][B][STO▶]

To calculate the mass, for example, type the integral



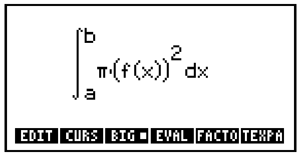
Then, press [ENTER][↱][→NUM] to obtain a numerical value for the integral. The mass is m = 5.56740083556.

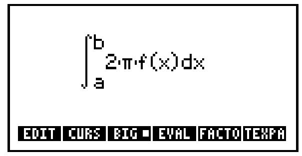
The following screens show the integrals you need to type to produce the first moment M_y, and moments of inertia I_y and I_x, respectively:



The corresponding values are M_y = 16.9252199715, I_y = 53.2640598494, and I_x = . Also, $\bar{x} = M_y/m$ = 3.04005773455, $k_y = (I_y/m)^{1/2}$ = 3.09307811219, and $k_x = (I_x/m)^{1/2}$ = 0.985351404481.

The volume and exterior area of the solid of revolution are calculated using the following integrals:

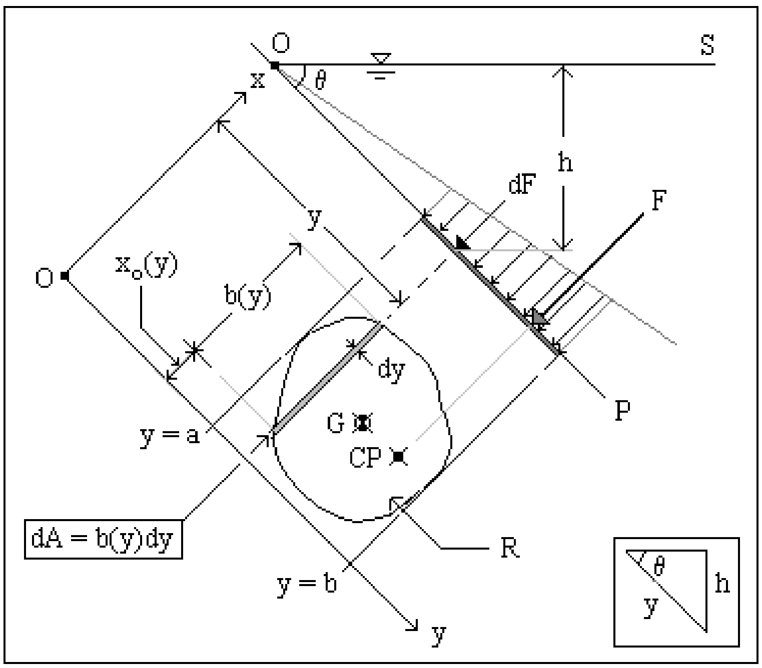




The results are V = 12.0257015609, and A_s = 17.2872588354.

Hydrostatics: force over a flat surface submerged in a liquid

Consider the flat surface R located along the plane OP inclined by an angle θ with respect to the horizontal free surface OS of a liquid at rest.



The pressure in a liquid at rest is given by

$$p = p_0 + \gamma \cdot h,$$

where p_0 is the pressure at the free surface γ is the specific weight of the liquid (weight/volume), and h is the depth measured from the free surface. If the free surface OS is open to the atmosphere, and we use gage pressures, $p_0 = 0$, and

$$p = \gamma \cdot h.$$

Since pressure depends on depth only, a differential of force dF acting on the differential of area

$$dA = b(y)dy$$

of the surface R will be given by

$$dF = pdA = \gamma \cdot h \cdot dA = \gamma \cdot h \cdot b(y) \cdot dy = \gamma \cdot y \cdot \sin\theta \cdot b(y) \cdot dy,$$

where the relationship

$$h = y\cdot\sin\theta$$

has been used. The force on the surface will be calculated through the integral

$$F = \int_R dF = \int_R p\cdot dA = \int_R \gamma\cdot h\cdot dA = \gamma\cdot\sin\theta\cdot\int_a^b y\cdot dA = \gamma\cdot\sin\theta\cdot\int_a^b y\cdot b(y)\cdot dy.$$

To find the point of application of the force we can take the moment of the differential of force dF with respect to the x and y axes as

$$dM_x = y\cdot dF = y\cdot p\cdot dA = \gamma\cdot h\cdot y\cdot dA = \gamma\cdot h\cdot y\cdot b(y)\cdot dy = \gamma\cdot y^2\cdot\sin\theta\cdot b(y)\cdot dy,$$

and

$$dM_y = [x_o(y)+b(y)/2]\cdot dF = [x_o(y)+b(y)/2]\cdot p\cdot dA = [x_o(y)+b(y)/2]\cdot\gamma\cdot h\cdot dA = [x_o(y)+b(y)/2]\cdot\gamma\cdot y\cdot\sin\theta\cdot dA$$

$$dM_y = [x_o(y)+b(y)/2]\cdot\gamma\cdot y\cdot\sin\theta\cdot b(y)\cdot dy$$

where $x_o(y)$ is the distance from the y-axis to the left edge of the differential of area, and b(y) is the width of the differential of area. The moments with respect to the x- and y-axes are given, respectively, by the following integrals

$$M_x = \int_R y\cdot dF = \int_R y\cdot p\cdot dA = \int_R \gamma\cdot y\cdot h\cdot dA = \gamma\cdot\sin\theta\cdot\int_a^b y^2\cdot dA = \gamma\cdot\sin\theta\cdot\int_a^b y^2\cdot b(y)\cdot dy.$$

and

$$M_y = \gamma\cdot\sin\theta\cdot\int_a^b [x_o(y)+\frac{1}{2}\cdot b(y)]\cdot y\cdot dA = \gamma\cdot\sin\theta\cdot\int_a^b [x_o(y)+\frac{1}{2}\cdot b(y)]\cdot y\cdot b(y)\cdot dy.$$

The point of application of the total force F is known as the center of pressure of the surface (point CP in figure above). Its coordinates are given as the arm of the force with respect to each axes that produce the same moments Mx and My. So, if the coordinates of CP are (x_{CP}, y_{CP}) we can write:

$$x_{CP} = M_y/F, \text{ and } y_{CP} = M_x/F.$$

Example 1 – Hydrostatic force on a triangular shape

Consider the triangular-shaped region located along the plane OP inclined by an angle θ from the free surface OS of a liquid at rest. The dimensions of the triangular surface and its location with respect to the x- and y-axes are indicated in the sketch. Find expressions for the force F, and the moments M_y and M_x produced by the hydrostatic pressure distribution on the triangular surface. Also find the coordinates of the center of pressure x_{CP} and y_{CP}.

- To find an expression for $x_o(y)$ we use the coordinates of points A and B. The slope of the line AB can be found from

$$m_{AB} = (y_B-y_A)/(x_B-x_A) = (a+H-a)/(d-d-b_1) = -H/b_1.$$

The equation of a straight line going through point A with slope m_{AB} is $y – y_A = m_{AB}(x-x_A)$, or

$$y = y_A + m_{AB}(x-x_A) = a–(H/b_1)(x-d-b_1).$$

Using the calculator you can isolate x as follows:

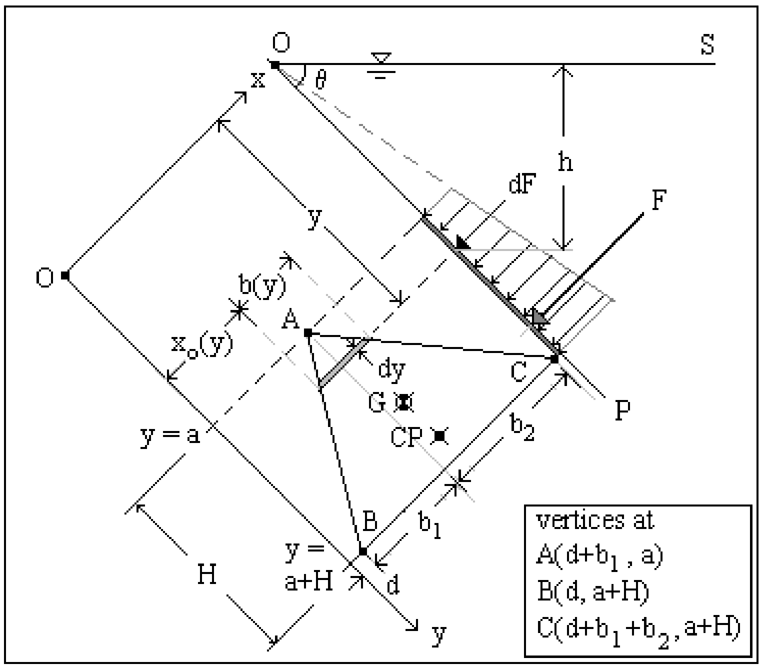
‘y=a-(H/b1)*(x-d-b1)’[ENTER] ‘x’ [ENTER] [↰][S.SLV][ISOL]

The result is: ‘x = ((d+b1)*H+(b1*a-y*b1))/H’

To define this result as the function ‘xo(y) = ((d+b1)*H+(b1*a-y*b1))/H’, use the equation writer as follows:

[▼][▼][⇦][ALPHA][↰][O] [↰][( )] [ALPHA][↰][Y] [ENTER] [↰][DEF]

This operation creates the variable [xo].



Similarly, the slope of line AC can be found from

$$m_{AC} = (y_C-y_A)/(x_C-x_A) = (a+H-a)/(d+b_1+b_2-d-b_1) = H/b_2,$$

and the equation of the line AC is

$$y = y_A+ m_{AC}(x-x_A) = a+(H/b_2)(x-d-b_1).$$

Let’s isolate x = x(y) by using the HP 49 G calculator:

‘y=a+(H/b2)*(x-d-b1)’[ENTER] ‘x’ [ENTER] [↰][S.SLV][ISOL]

The result is: `‘x = -(((a-y)*b2-(d+b1)*H)/H’`.

This result represents an outer value of x that we can define as xf(y). To define this function use:

[▼][▼][⇦][ALPHA][↰][F] [↰][( )] [ALPHA][↰][Y] [ENTER] [↰][DEF]

This operation creates the variable [xf].

- The width of the element of area b(y) is by definition b(y) = xf(y) – xo(y), thus, we can define b(y) with the HP 49 G calculator by using:

‘xf(y) – xo(y)’ [ENTER][EVAL] [↱][‘] [ALPHA][↰][B] [↰][( )] [ALPHA][↰][Y] [ENTER] [▶][↱][=]

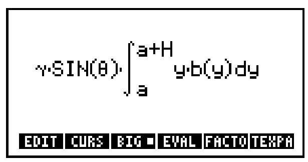
This results in `‘b(y) = -(((a-y)*b1+(a-y)*b2)/H)’`.

Use [↰][DEF] to define the function b(y).

Having defined xo(y) and b(y) we can proceed to calculate the force F, moments M_y and M_x, and coordinates of the center of pressure x_{CP}, y_{CP}, as follows:

Note: The Greek letter γ is available in the HP 49 G calculator’s character set by using [↱][CHAR], selecting γ, and pressing [ECHO1].

- Force:



Press [ENTER] [↱][EVAL]. After about 20 seconds you get the following result:

`‘((2*γ*H^2+3*γ*a*H)*b1+(2*γ*H^2+3* γ*a*H)*b2)*SIN(θ)/6’`

Use [↱][ALG][FACTO] to get `‘SIN(θ)*H*γ*((2*H+3*a)*b1+(2*H+3*a)*b2)/(3*2)’`.

In the latter expression you can recognize another common factor (2*H+3*a) that has not yet been factored out. You can factor it out by using the equation writer as follows:

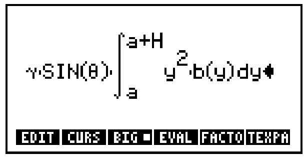
[▼][▼][▼][▼][▼] [▶][▶][▶] [▲][▲][▲][▲][▲] [FACTO] [ENTER]

The result is now `‘SIN(θ)*H*γ*((b1+b2)*(2*H+3*a))/(3*2)’, i.e..`

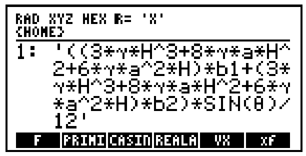
$$F = \frac{1}{6}\cdot\gamma\cdot H\cdot\sin\theta\cdot(b_1+b_2)\cdot(2\cdot H+3\cdot a).$$

Save this result into variable F by using [↱][‘][ALPHA][F][STO▶]

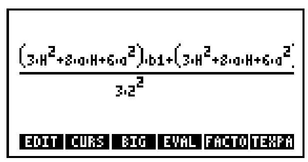
Moment about the x axis:



Press [ENTER] [↱][EVAL]. After about 40 seconds you get the following result:



To factorize this expression use [↱][ALG][FACTO] . Press [▼] to activate the equation writer. If the option [BIG■] is selected, press the corresponding button to de-select it. This will let you see the current expression in a smaller font. There is a common factor that has not been factored out. To move about the equation writer screen, use the following: [▼][▼][▼][▼]. This will change the cursor to a rectangular shape that you can move from term to term. Press [▶] twelve times to place the rectangular cursor over the term b1. The screen should look like this:

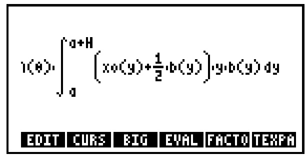


This screen lets you see the common factor that is still distributed in the expression. This common factor is $(3 \cdot H^2+8 \cdot a \cdot H+6 \cdot a^2)$. To factor it out, use: [▲][▲][▲][FACTO]. Press [ENTER]. The resulting expression can be translated in paper as

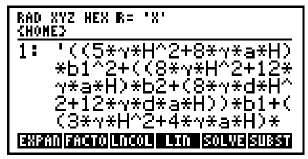
$$M_x = \frac{1}{12} \cdot \gamma \cdot H \cdot \sin\theta \cdot (b_1 + b_2) \cdot (3 \cdot H^2 + 8 \cdot a \cdot H + 6 \cdot a^2).$$

Save this result in variable Mx using: [↱][‘][ALPHA][M] [ALPHA][↰][X] [STO▶].

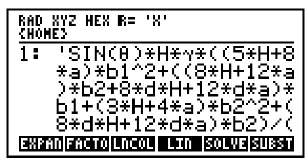
- Moment about the y axis:



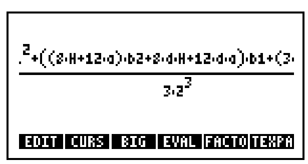
Press [ENTER] [↱][EVAL]. This calculation will take more than a minute since it involves multiplying out and integrating a more complicated expression. The beginning of the resulting expression is presented in the following screen:



To factorize this expression use [↱][ALG][FACTO]. The result is now:



In this last result we can see some common factors still not factored out, e.g., *(8*H+12*a)* and *(8*d*H+12*d*a)*. Press [▼] to activate the equation writer. (If the option [BIG■] is selected, press the corresponding button to de-select it. This will let you see the current expression in a smaller font. To move about the equation writer screen, use the following: [▼][▼][▼][▼]. This will change the cursor to a rectangular shape that you can move from term to term. Press [▶] fourteen times to place the rectangular cursor over the term 8. The screen should look like this:



Now, press [▲][▲][▲] to select the expression (8·H+12·a)·b2+8·d·H+12·d·a. Press [FACTO] to factor this expression to (b2+d)· (8·H+12·a). Now, press [▼][▶] to highlight the term (8·H+12·a). Press [FACTO] to convert this expression to (2·H+3·a)·4. Next, press [▶][▶] to highlight the last term in the numerator: (8·d·H+12·d·a)·b2. Press [FACTO] to obtain for this last term b2·d· (8·H+12·a). We can factor this term even further by using [▼][▶][▶][FACTO] to highlight the term (8·H+12·a) and factor it out to (2·H+3·a)·4. Press [▼] until you obtain the rectangular cursor. Then, use the left- and right-arrow keys to move about the expression. The expression can be translated into paper as follows:

$$\frac{SIN(\theta)\cdot H\cdot\gamma\cdot((5\cdot H+8\cdot a)\cdot b1^2+(b2+d)\cdot(2\cdot H+3\cdot a)\cdot b2^2+b2\cdot d\cdot(2\cdot H+3\cdot a)\cdot 4)}{3\cdot 2^2}$$

There is still the factor (2H+3a) that can be factored out of the last two terms in the numerator. Press [▼] until the rectangular cursor is available. Then move the cursor on top of the term b2 contained in (b2+d). Next, press [▲][▲][▲][▲][FACTO]. This result in the expression:

$$\frac{SIN(\theta)\cdot H\cdot\gamma\cdot(b1+b2)\cdot((5\cdot H+8\cdot a)\cdot b1+(3\cdot H+4\cdot a)\cdot b2+8\cdot d\cdot H+12\cdot d\cdot a)}{3\cdot 2^2}$$

This is an improvement as we were able to identify the factor (b1+b2), however, within the second set of parentheses in the numerator we still have some factoring to do, particularly, in the last two terms. Press [▼] until the rectangular cursor is available. Then move the cursor on top of the term 8 contained in 8·d·H Press [▲][▲][▲] until the term 8·d·H is highlighted. Next, press [↱][▶] to highlight the last two terms, and press [FACTO]. The highlighted term is converted to d· (2···H+3a)4. The entire expression now looks like this:

$$\frac{SIN(\theta)\cdot H\cdot\gamma\cdot(b1+b2)\cdot((5\cdot H+8\cdot a)\cdot b1+(3\cdot H+4\cdot a)\cdot b2+d\cdot(2\cdot H+3\cdot a)\cdot 4)}{3\cdot 2^2}$$

Press [ENTER] and save the result into variable M_y, by using:

[↱][‘][ALPHA][M] [ALPHA][↰][Y] [STO▶].

Press [VAR]. You should have in your soft-menu key labels the following keys:

[My][Mx][F].

- Coordinates of the center of pressure:

To calculate the coordinate $x_{CP} = M_y/F$ use:

[My][F] [÷] [↱][ALG][FACTO]

After about 30 seconds the calculator returns the result:

‘((5*H+8*a)*b1+((3*H+4*a)*b2+(8*d*H+12*d*a)))/(8*H+12*a)’.

This result can be factored even more using the equation writer to make it look like this:

$$\frac{(5\cdot H+8\cdot a)\cdot b1+(3\cdot H+4\cdot a)\cdot b2+d\cdot(2\cdot H+3\cdot a)\cdot 4}{(2\cdot H+3\cdot a)\cdot 4}$$

This latter result suggest that we can write:

$$x_{CP} = d + \frac{(5 \cdot H + 8 \cdot a) \cdot b1 + (3 \cdot H + 4 \cdot a) \cdot b2}{4 \cdot (2 \cdot H + 3 \cdot a)}.$$

To calculate the coordinate $y_{CP} = M_x/F$ use:

[Mx   ][F   ] [÷] [↱][ALG][FACTO]

After about 20 seconds the calculator returns the result:

```
'(3*H^2+8*a*H+6*a^2)/(4*H+6*a)'.
```

The denominator has a common factor of 2, but no other simplification is possible, so the result is:

$$y_{CP} = \frac{3 \cdot H^2 + 8 \cdot a \cdot H + 6 \cdot a^2}{2 \cdot (2 \cdot H + 3 \cdot a)}.$$

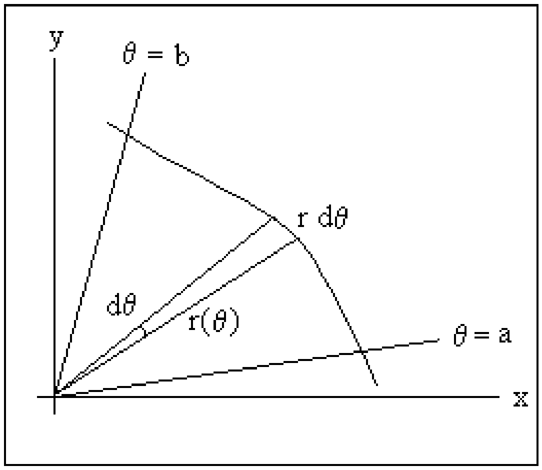
Now, check these results by hand.

Note: Just kidding!

Mathematics: area in polar coordinates

The figure below illustrates a region R described by { $0 < r < f(\theta), a < \theta < b$ }. The quasi-triangular infinitesimal element of area limited by the angle dθ and the curve has an area

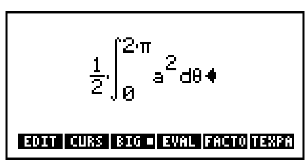
$$dA = ½\ (\text{base}) \cdot (\text{height}) = ½\cdot (r\cdot d\theta)\cdot (r) = ½\cdot r^2\cdot d\theta = ½\cdot [f(\theta)]^2\cdot d\theta.$$



Therefore, the area of the region is given by

$$A = \int_R dA = \frac{1}{2}\cdot\int_{\theta=a}^{\theta=b}[f(\theta)]^2\cdot d\theta.$$

Example 1 – Calculate the area of the region R = { $0 < r < a, 0 < \theta < 2\pi$ }, i.e., the area of the circle of radius r = a centered at the origin. Type in the integral:



Press [▲][▲][▲][EVAL] to get the result $a^2\cdot\pi$.

Fluid dynamics: calculating discharge in pipe for laminar flow

The figure below shows the profile of laminar flow velocity as a function of the radial distance *r* in a pipe.

The velocity distribution is given by the expression,

$$v(r) = v_c[1-(r/r_o)^2],$$

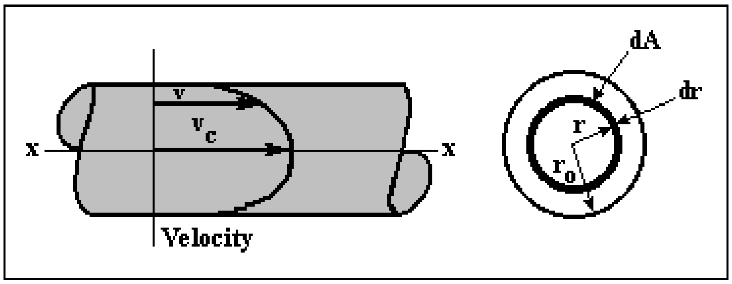
where v_c is the centerline velocity and r_o is the radius of the pipe.

We can use this expression to obtain the discharge (volumetric flow) in the pipe by using the definition

$$Q = \int_R v \cdot dA.$$

Because the velocity distribution in a pipe depends on the radial distance only, we can use an element of area consisting of a ring of thickness dr and length 2πr, thus, the area is

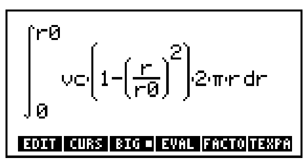
$$dA = 2\pi r dr.$$



With this element of area, the discharge is calculated, in general, as

$$Q = \int_0^{r_o} v(r) \cdot 2 \cdot \pi \cdot r \cdot dr.$$

For the specific case of a laminar flow velocity distribution, you will need to set up the integral:



Then, press [ENTER][↱][EVAL]. The result is 'vc*r0^2*π/2'.

The mean velocity is defined as V = Q/A, with $A = \pi \cdot r_o^2$, then $V = v_c/2$ for laminar flow.

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