Working with Surfaces with the HP 49 G Calculator

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Working with surfaces with the HP 49 G

Surfaces in space can be represented implicitly as

$$f(x,y,z) = 0,$$

or explicitly as

$$z = g(x,y).$$

The latter can be converted into an implicit representation by writing,

$$f(x,y,z) = z - g(x,y) = 0.$$

Surfaces described by z = g(x,y) can be plotted in the HP 49 G calculator by using *Fast3D* or *Wireframe* graphs. Examples of these type of graphics are presented also in Chapter 11.

A parametric representation of a surface in terms of the two parameters u and v is,

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}.$$

Parametric representations of surfaces can be plotted in the HP 49 G calculator by using *Pr-Surface* graphs.

<u>*Example 1*</u> – Plot the surface z = sin(x)cos(y) using a *Fast3D* graph.

- Press [↰][2D/3D], simultaneously to access to the PLOT SETUP window.

➢ Change `TYPE` to `Fast3D.`

➢ Press [▼] and type 'SIN(X)*COS(Y)' [OK].

➢ Make sure that 'X' is selected as the `Indep:` and 'Y' as the `Depnd:` variables.

➢ Press [NXT][OK] to return to normal calculator display.

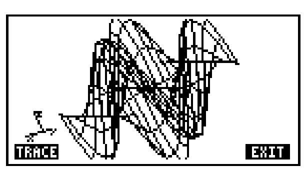
- Press [↰][WIN], simultaneously, to access the PLOT WINDOW screen.

➢ Change the plot window ranges to read:

```
X-Left:-3.15     X-Right:3.15
Y-Near:-3.15     Y-Far: 3.15
Z-Low: -1        Z-High: 1

Step Indep: 15  Depnd: 12
```

➢ Press [ERASE][DRAW] to draw the three-dimensional surface. The following figure shows one view of the FAST3D plot for this example:



- Press [EXIT] to return to the PLOT WINDOW environment.
- Press [CANCL] to return to PLOT WINDOW.
- Press [ON], or [NXT][OK], to return to normal calculator display.

Example 2 – Plot the surface represented parametrically by x = x(X,Y) = X sin Y, y = y(X,Y) = x cos Y,

Pr-Surface (parametric surface) plots are used to plot a three-dimensional surface whose coordinates (x,y,z) are described by x = x(X,Y), y = y(X,Y), z=z(X,Y), where X and Y are independent parameters.

Note: The equations x = x(X,Y), y = y(X,Y), z=z(X,Y) represent a parametric description of a surface. X and Y are the independent parameters. Most textbooks will use (u,v) as the parameters, rather than (X,Y). Thus, the parametric description of a surface is given as x = x(u,v), y = y(u,v), z=z(u,v).

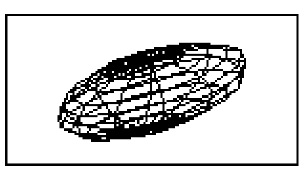
For example, to produce a Pr-Surface plot for the surface x = x(X,Y) = X sin Y, y = y(X,Y) = X cos Y, z=z(X,Y)=X^2, use the following:

- Press [↰][2D/3D], simultaneously to access to the PLOT SETUP window.
- Change `TYPE` to `Pr-Surface.`
- Press [▼] and type '{X*SIN(Y) X*COS(Y) X^2}' [OK].
- Make sure that 'X' is selected as the `Indep:` and 'Y' as the `Depnd:` variables.
- Press [NXT][OK] to return to normal calculator display.

- Press [↰][WIN], simultaneously, to access the PLOT WINDOW screen.
- Change the plot window ranges to read:

```
X-Left:-1 X-Right:1
Y-Near:-1 Y-Far: 1
Z-Low: -1 Z-High:1
XE: -8   YE:-8   zE:-8
```
Step Indep: 15 Depnd: 15

- Press [ERASE][DRAW] to draw the three-dimensional surface. The result is a tree-dimensional surface that looks like a symmetric watermelon. Ideally, this should be the plot of a sphere. The distortion occurs because of the rectangular shape of the calculator's screen. By the way, it is going to take the calculator a couple of minutes to finish the plot. So, be patient. Here is what the surface looks like:



- Press [EDIT][NXT][LABEL][MENU] to see the graph with labels and ranges.

- Press [NXT][NXT][PICT][CANCL] to return to the PLOT WINDOW environment.
- Press [ON], or [NXT][OK], to return to normal calculator display.

Area of a parametric surface

The areas A of a surface S: **r**(u,v) is defined by the double integral

$$A = \iint_R dA = \iint_R |\, \mathbf{r}_u \times \mathbf{r}_v \,|\, du \cdot dv,$$

where

$$\mathbf{r}_u = \partial\mathbf{r}/\partial u, \qquad \mathbf{r}_v = \partial\mathbf{r}/\partial v,$$

and R is the region of the u-v plane corresponding to the surface.

The expression

$$dA = |\, \mathbf{r}_u \times \mathbf{r}_v \,|\, du \cdot dv,$$

is called the element or differential of area of the surface.

<u>*Example 1*</u> -- Consider the surface S: **r** = **r**(u,v) = [R·sin u·cos v, R·sin u·sin v, R·cos u;], 0<u<π/2, 0<v<π. To obtain the integrand $|\mathbf{r}_u \times \mathbf{r}_v|$ we use the program *VDeriv* within sub-directory VCALC, as well as the HP 49 G function CROSS, as follows:

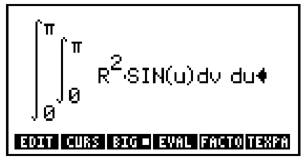
['R*SIN(u)*COS(v)' 'R*SIN(u)*SIN(v)' 'R*COS(u)'] [ENTER] 'r' [STO▶]	Store **r**
[VDeri] 'r' [▼] 'u' [ENTER]	Calculate $\mathbf{r}_u$
[VDeri] 'r' [▼] 'u' [ENTER]	Calculate $\mathbf{r}_v$
[↰][MTH][VECTR][CROSS] [↰][ABS]	Calculate $|\mathbf{r}_u \times \mathbf{r}_v|$
[↱][EVAL][↱][ALG][FACTO][↱][TRIG][NXT][NXT][TRIG]	Simplify last result

The result is 'R^2*SIN(u)'.

To create the double integral required for calculating the area, we will use the equation writer as follows:

[▼]	Enter eq. writer, expression is highlighted.
[↱][∫][0] [▶] [↰][π]	Place expression in integral, enter limits
[▶][▶] [ALPHA][↰][V]	Enter variable of integration for inner integral
[▲][▲][↱][∫][0] [▶] [↰][π][÷][2]	Select/place expression in integral, enter limits
[▶][▶] [ALPHA][↰][U]	Enter variable of integration for outer integral

The integral should look like this:



The integral is calculated by using [▲][▲][EVAL]. The result is '2*R^2*π', which is the area of the hemisphere defined by the surface S.

Area of a surface z = g(x,y)

A surface S represented by z = g(x,y) can also be written as

$$\mathbf{r}(u,v) = u\,\mathbf{i} + v\,\mathbf{j} + g(u,v)\,\mathbf{k},$$

with

$$r_u = \mathbf{i} + g_u\mathbf{k},\quad r_v = i + g_v\mathbf{k},$$

and

$$|\,\mathbf{r}_u \times \mathbf{r}_v\,|^2 = 1 + g_u^{\,2} + g_v^{\,2} = 1 + (\partial g/\partial u)^2 + (\partial g/\partial v)^2.$$

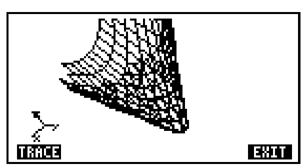
Replacing u = x and v = y in this result we have:

$$A = \iint_{S^*} dA = \iint_{S^*} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\cdot dy,$$

where S* is the projection of S on the x-y plane.

<u>*Example 1*</u> – Determine the side area of the cone described by S: { $z = (x^2+y^2)^{1/2}$, 0<z<1 }.

The surface can be plotted in the HP 49 G calculator using a Fast3D graph to look as follows:

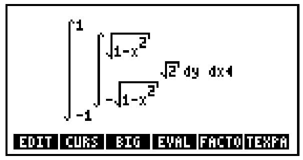


The projection of the surface on the x-y plane is obtained by replacing z = 1 in the expression $z = (x^2+y^2)^{1/2}$, resulting in $x^2+y^2 = 1$, or S* = { $-(1-x^2)^{1/2} < y < (1-x^2)^{1/2}$, -1 < x < 1}. To put together the integral use the following:

'√ (x^2+y^2)'[ENTER][ENTER]	Enter g(x,y), make extra copy
'x' [ENTER][↦][∂] [2] [y^x][ENTER] [↦][EVAL]	Calculate $(\partial g/\partial x)^2$
[▶]	Swap levels 1 and 2, g(x,y) is now in level 1
'y' [↦][∂][↤][x^2][↦][EVAL]	Calculate $(\partial g/\partial y)^2$
[+]	Calculate $(\partial g/\partial x)^2 + (\partial g/\partial y)^2$
[1] [+][√x]	Calculate $[1+ (\partial g/\partial x)^2 + (\partial g/\partial y)^2]^{1/2}$

The result is '√2'.

Now, put together the following integral:



Finally, press [▲][▲][EVAL], within the equation writer, to obtain the result '√2*π'.

Surface integrals

As a line integral is a generalization of the univariate integral, the surface integral is a generalization of the double integral. Let S be a surface represented by **r**(u,v), with dA = | $\mathbf{r}_u \times \mathbf{r}_v$ | du·dv, then the surface integral of f(x,y,z) over S is calculated as

$$\iint_S f(x,y,z)dA = \iint_S f[x(u,v),y(u,v),z(u,v)]\cdot|\mathbf{r}_u \times \mathbf{r}_v| \, du\cdot dv.$$

If S is represented in the form z = g(x,y), then

$$\iint_{S^*} f(x,y,z)\cdot dA = \iint_{S^*} f[x,y,g(x,y)]\cdot\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\cdot dy,$$

Orientation of a surface

Let S be a smooth surface. At any point P on S we can select a unit normal vector **n**. The direction of **n** is called the positive normal direction of S at P. There are, obviously, two possibilities in choosing **n** for any given surface in space. The two expressions for the normal vector can be obtained from

$$\mathbf{n} = \pm\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

A smooth surface is said to be *orientable* if the positive normal direction, when given at an arbitrary point P of S, can be continued in a unique and continuous way over the entire surface. The external surface of a sphere, for example, will be an orientable surface. Classical examples of no-orientable surfaces are the Moebius strip, and Klein's bottle.

If a smooth surface is orientable, then we may orient the S by choosing one of the two possible directions of the normal vector.

Evaluating surface integrals over a parametric surface

Depending on the sign selected for the normal vector **n**, the integral of a function f(x,y,z) over a surface **r**(u,v) can be calculated as

$$\iint_S f(x,y,z)dxdy = \pm\iint_R f[x(u,v),y(u,v),z(u,v)]\cdot J(\frac{x,y}{u,v})\cdot dudv,$$

Where R is the region corresponding to S in the u-v plane.

<u>*Example 1*</u> -- Determine the value of the integral, $\iint_R[x^2+y^2+(z-a)^2]^{-1/2}dA$, on the hemisphere S:$x^2+y^2+z^2= a^2$, z>0. [Hint: represent S by **r**(u,v) = a sin u cos v **i** + a sin u sin v **j** + a cos u **k**.]

Start by putting together the Jacobian, using the program JACOBIAN developed earlier:

[‘a*SIN(u)*COS(v)’ ‘a*SIN(u)*SIN(v)’ ‘a*COS(u)’] [ENTER]	Enter vector [‘x(u,v)’ ‘y(u,v)’]
[‘u’ ‘v’][ENTER] [JACOB]	Enter vector [‘u’ ‘v’]
[↱][EVAL][↱][ALG][FACTO][↱][TRIG][NXT][NXT][TRIG]	Simplify the expression

The result is ‘a^2*COS(u)*SIN(u)’. Next, evaluate the function in terms of u,v, as follows:

‘1/√ (x^2+y^2+(z-a)^2)’	[ENTER]	Enter expression for function
‘x = a*SIN(u)*COS(v)’	[ENTER][↱][ALG][SUBST]	Substitute x = x(u,v)
‘y = a*SIN(u)*SIN(v)’	[ENTER][↱][ALG][SUBST]	Substitute y = y(u,v)
‘z = a*COS(v)’	[ENTER][↱][ALG][SUBST]	Substitute z = z(u,v)
[↱][EVAL][↱][ALG][FACTO][↱][TRIG][NXT][NXT][TRIG]		Simplify the expression

The result is ‘1/(√ -(2*(COS(u)-1)*ABS(a))’. Multiply the two terms and simplify the result, by using:

[×][↱][EVAL].

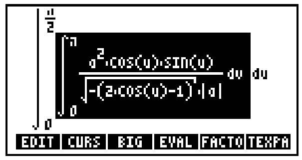
Next, put together the following double integral using the equation writer:

[▼]	Enter equation writer, expression is highlighted.
[↱][∫][0] [▶] [↰][π]	Place expression in integral, enter limits
[▶][▶] [ALPHA][↰][V]	Enter the variable of integration for inner integral
[▲][▲][↱][∫][0] [▶] [↰][π][÷][2]	Select and place expression in integral, enter limits
[▶](12 times) [ALPHA][↰][U]	Enter the variable of integration for outer integral

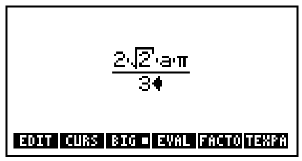
To evaluate the integral, step-by-step, use:

[▲][▲]	Select entire double integral
[▼][▼]	Select lower limit of outer integral, change cursor
[▶][▶][▶][▶][▶]	Move cursor to integrand expression
[▲][▲][▲][▲][▲]	Select inner integral

The screen should look like this:



Then, press [EVAL]. Highlight the remaining integral, and press [EVAL] again. Because the calculator does not know that a is a real number, it will request to change settings to Complex, accept the change as requested. The result is:



Considering the complicated expression that we are integrating, you should allow the calculator a few minutes to achieve the result.

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