# Real and Complex Number Operations with the HP 49 G Calculator


**By**

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# Calculations with real numbers

In this document we introduce the Calculator Algebraic System and show simple operations with real and complex numbers.

## What is the CAS in the HP 49 G calculator?

*CAS* stands for *Calculator Algebraic System*. This is the ROM-based software that controls the algebraic (symbolic) operations within the calculator. The CAS presents a variety of options that can be set by the user by pressing the [MODE] key, and then pressing the [CAS] soft key. This generates a screen called *CAS MODES*, which lets you set the CAS options. To see an explanation for the different options use the arrow keys to move to the different fields in the *CAS MODES* screen. The description of a given field is shown at the bottom of the calculator display.

### CAS options

The different options available in the *CAS MODES* screen are:

- *Indep var*: 'X' -- Enter independent variable name. By default 'X' is the preferred independent variable. This value is stored in a variable called VX, which appears in the HOME directory of the calculator. Many of the algebraic operations in the HP 49 G use this default independent variable.
- *Modulo*: 3 – Enter modulo value. The modulo value is used in operations with polynomials. The default value is 3.
- *Numeric* – replace constants by values? When this option is selected, the results will be evaluated to numerical, rather than symbolic, expressions.
- *Approx* – Perform approx calculations? Selecting this option will produce numerical results in decimal format rather than as fractions for rational numbers or algebraic expressions for other type of results.
- *Complex* – Allow complex numbers? This option allows for complex numbers to result from certain operations.
- *Verbose* – Display calculus information? This option allows the calculator to provide information relevant to some calculus operations (derivatives, integrals, etc.).
- *Step/Step* – Perform operations step by step? Selecting this option allows the calculator to show intermediate steps in operations such as integration, matrix inversion, solution of linear equations, etc.
- *Incr Pow* – Increasing polynomial ordering? Selecting this option will show polynomial terms in order of increasing power rather than the default setting of decreasing power order.
- *Rigurous* – Don't simplify |X| to X? If selected, will keep absolute value expressions from defaulting to the positive value (the principal value).
- *Simp* – Simplify non rational expr? Needs no explanation.

Many of these settings are requested by the calculator when a particular result requires it. For example, if you try to get the square root of a negative number while in REAL (rather than COMPLEX) mode, the calculator will request that you change the mode to COMPLEX. If you refuse, the calculator aborts the operation and provides a square root error: <!> √ Error: Mode switch cancelled

### Checking current CAS settings

To check the current CAS settings you need to just look at the top line in the calculator display in normal operation. For example, you may see the following setting:

RAD XYZ DEC **R** = 'X'

This stands for RADians for angular measurements, XYZ for Rectangular (Cartesian) coordinates, DECimal number base, **R**eal numbers preferred, = means "exact" results, and 'X' is the value of the default independent variable.

Another possible listing of options could be

DEG R∠Z HEX **C** ~ 't'

This stands for DEGrees as angular measurements, R∠Z for Polar coordinates, HEXagesimal number base, **C**omplex numbers allowed, ~ stands for "approximate" results, and 't' as the default independent variable.

### Checking the calculator mode

When in RPN mode the different levels of the stack are listed in the left-hand side of the screen. When the ALGEBRAIC mode is selected there are no numbered stack levels, and the word ALG is listed in the top line of the display towards the right-hand side.

## Real number calculations

This section shows some examples of calculations with real numbers. Some of these calculations may, under the right conditions, produce a complex result. We will show you how to deal with such situations, but a complete coverage of complex numbers is not presented until later in this book. Set number format to standard. Also, within CAS select exact, as opposite to approximate, and real as opposite to complex.

[MODE] [▼] (toggle [+/-] key until Std shows in the field)

[ CAS ] [▼][▼] (toggle [+/-] key, or press the [✓CHK] key, until the option Approx is unchecked.)

[▶] (toggle [+/-] key, or press the [✓CHK] key, until the option Complex is unchecked.)

Press [OK][OK] to return to normal calculator display.

The calculations with real numbers will be presented as exercises for you to try, as follows:

- **Exponential**, $e^x$, e.g., [1][↰][$e^x$] results in EXP(1). The reason for this result is that we selected the *exact* option, which results in algebraic formulae for the results. To obtain a numeric result, we can use [↱][→NUM]. The result is 2.7182… When the *approximate* setting is selected, the numerical result is automatic.

- **Natural logarithm**, LN, e.g., [5][↱][LN] results in LN(5). Press [↱][→NUM]., to obtain the numerical value: 1.60943…

Note: the exponential and natural logarithm functions are inverse functions of each other, i.e.,

if $y = ln\ x$, then, $x = e^y$,

and vice versa.

 

- **Power**, $y^x$, e.g., $5^{-2}$, [5][ENTER][ 2][+/-] [$y^x$] results in an error message: ^ Error: Negative integer. The reason is that, with the *exact* mode selected for results, all numbers entered without a decimal part are considered integers. When raising a number to a integer power, the calculator will try to multiply the base an integer number of times to calculate the power. This works fine for positive integers, but not for negative integers. To work with negative integer powers, we need to select the *approximate* mode:

[MODE] [ CAS ] [▼][▼][▼][+/-][OK][OK]

Try the operation again: [5][ENTER][2][+/-] [$y^x$], resulting in .04.
An alternative sequence for this operation is: [5][SPC][2][+/-][$y^x$].

The calculation of a power in the approximate method uses logarithms and anti-logarithms to calculate the power, i.e., $a^x = exp(x \cdot ln\ a)$.

- **Square power**, $x^2$, e.g., [5][↰][ $x^2$] results in 25.

- **Square root**, √x, e.g., [8] [√x] results in 2·√2, in the exact mode. The numerical value, obtained with [↱][→NUM], is 2.828427…

- **x-th root of y**, $^x\sqrt{\ }$ y, e.g., [2][7][+/-][SPC][3] [↱][$^x\sqrt{\ }$ y] results in the value -3.

Let's check out what happened if the exact mode is selected:

[MODE] [ CAS ] [▼][▼][+/-][OK][OK]
[2][7][+/-][SPC][3] [↱][$^x\sqrt{\ }$ y]

In this case, you are asked to select the complex mode before the calculator provides a result. This is so because we now are using the exact mode. In the approximate mode condition, the calculator selects, by default, the principal root. At this point, if you were to select the option NO for the complex mode, then an error will be reported. Try it: [▼][OK]. The result is the message: XROOT Error: Mode switch cancelled.

Repeat the exercise, but this time accept the change of mode to complex:

[2][7][+/-][SPC][3] [↱][$^x\sqrt{\ }$ y] [OK]

Now, since the angle measure was set to degrees, you get asked if you want to change that mode to radians. If you refuse the change, you will get the same error obtained above when refusing to change to complex. Thus, press [OK], to obtain the result:

'3*EXP(*i**π/3)',

which is a complex number. (More about complex numbers later).

- **Powers of 10**, $10^x$, e.g., [3][.][1][↰][$10^x$] will produce the value 1258.92541179.

- **Base-10 logarithm**, LOG, e.g., [5][↱][LOG] produces the value LOG(5) if the CAS is set to EXACT mode. Press [↱ ][→NUM]., to obtain the numerical value: 0.69897…

These two operations are inverse of each other, i.e., $10^{\log x} = x$, and $\log 10^x = x$.

## Modifying the display to obtain a textbook-like appearance

This frame refers to modifications that you can make to the display to change from the one-line algebraic expression, such as the result, '3*EXP(*i**π/3)', to a more textbook-like format, such as:

$$3 \cdot EXP\left(\frac{i \cdot \pi}{3}\right)$$

 

To change to a textbook-like appearance, use:

[MODE][DISP] [▼] [▼][▶], toggle the [+/-] key until the *Textbook* option is checked.  Then, press [OK][OK] to return to normal calculator display.

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## Changing from complex to real mode

Before continuing, I suggest we abandon the complex mode by using:

[MODE][ CAS ] [▼] [▼] [▶], toggle the [+/-] key until the complex option is unchecked.

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- **Trigonometric functions**, SIN, COS, TAN, can have their arguments in any of the valid angle units: degrees, radians, grads.  Inverse trigonometric functions, ASIN, ACOS, ATAN, will return the resulting angular measure in the units indicated in the upper left corner of the display. Try the following examples:

First, change mode to degrees: [MODE][▼][▼] , toggle the [+/-] key until Degrees is selected.  Then, press [OK].  The characters DEG are shown in the upper left corner of the display.   Try these exercises:

- [4][5] [SIN]  results in SIN(45), a symbolic result.  To get the numerical result, use [↱][→NUM].
- [2][6][5] [COS], results in COS(265). Numerical result, [↱][→NUM] results in –8.7155..E-2
- [2][5][+/-][TAN], results in –TAN(25).  Numerical result is –0.466307

More complicated expressions involving trigonometric functions can be built by using the Equation Writer, for example:

$$\frac{1+\tan\left(\frac{\pi}{3}\right)}{1+\tan\left(\frac{\pi}{4}\right)}$$

The angles in this expressions are given in radians.  To change the angle mode to radians we can access the CALCULATOR  MODES screen and change the Angle Measure feature, or we can just simply type RAD into register x and press [ENTER],i.e., [ALPHA][ALPHA][R][A][D][ENTER].  Then, enter the expression to be evaluated:

[EQW] [1][+][TAN] [↰][π] [÷] [3] [▶][▶][▶] [÷] [1][-][TAN] [↰][π][▶] [÷] [4]  [ENTER]

Using [↱][EVAL] you get (1+√3)/2.  If you have used instead [↱][→NUM], the result would have been 1.36602…

## Some functions accessed through the MTH menu

The MTH (MaTH) menu permits access to a number of functions as described below.  Press [↰][MTH] to activate this menu. The following folders (function menus) are available:

[VECTR][MATRX][LIST][ HYP ][REAL][BASE]

 

Pressing next reveals the menus:

[PROB][ FFT ][CMPLX][CONST]

Obviously, the folders [VECTR] and [MATRX] correspond to operations with vectors and matrices. These two topics will be discussed separately in later chapters. The folder [LIST] deals with lists of objects. The functions contained in the LIST folder are discussed in a later chapter in relation to some simple programming with User RPL. The [CMPLX] menu relates to complex number calculations which is the subject of the next chapter. The remaining MTH folders (i.e., HYP, REAL, BASE, PROB, FFT, and CONST) are described below.

**HYP menu**: hyperbolic functions, including SINH, COSH, TANH, and the inverse hyperbolic functions, namely, ASINH, ACOSH, ATANH. The [NXT] menu includes the functions EXPM and LNP1, with

$$EXPM(x) = \exp(x) - 1,$$

and

$$LNP1(x) = \ln(x+1).$$

Press [MTH] to return to the MTH menu.

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**Examples of the hyperbolic functions menu**

[2][.][5][SINH] : sinh 2.5 = 6.05020..
[2][.][5][COSH] : cosh 2.5 = 6.13228..
[2][.][5][TANH] : tanh 2.5 = 0.98661..
[2][ASINH][↱][ →NUM] : $\sinh^{-1}(2) = 1.4436…$
[2][ACOSH][↱][ →NUM]: $acos^{-1}(2) = 1.3169…$
[.][2][ATANH] : $\tanh^{-1}(0.2) = 0.2027…$
[NXT] provides the second menu:
[2][EXPM] [↱][ →NUM] => 6.38905….
[1][LNP1][↱][ →NUM] => 0.69314….

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**REAL menu**: functions applicable to real numbers mainly (some functions also applicable to complex numbers), such as:

- [ % ] : calculates the x percentage of y
- [ %CH] : calculates 100(y-x)/x, i.e., the percentage change
- [ %T ] : calculates 100 x/y
- [ MIN ] : minimum value of x and y
- [ MAX ]: maximum value of x and y
- [ MOD ]: y mod x = residual of y/x

Press [NXT] to get to the next menu set:

- [ ABS ] : calculates the absolute value, |x|
- [ SIGN ]: determines the sign of x, i.e., -1, 0, or 1.
- [MANT] : determines the mantissa of a number based on $\log_{10}$.
- [ XPON] : determines the power of 10 in the number
- [ IP ]: determines the integer part of a real number
- [ FP ]: determines the fractional part of a real number

Press [NXT] once more to get the final menu set:

- [ RND ] : rounds up y to x decimal places
- [ TRNC ] : truncate y to x decimal places
- [FLOOR] : closest integer that is less than or equal to x

 

- [ CEIL ] : closest integer that is greater than or equal to x
- [ D→R ] : converts degrees to radians
- [ R→D ] : converts radians to degrees.

Press [NXT] [MTH] to recover the MTH menu.

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***Examples of functions from the REAL menu***

| | |
|---|---|
| [2][0][ENTER][5][.][2][ % ] | : 5.2 % of 20 is 1.04 |
| [2][2][ENTER][2][5][ %CH] | : percentage change from 22 to 25 is 13.6363… |
| [5][0][0][ENTER][2][0][ %T ] | : 20 is the 4% of 500 |
| [2][+/-][ENTER][2][ MIN ] | : min(-2,2) = -2 |
| [2][+/-][ENTER][2][ MAX ] | : max(-2,2) = 2 |
| [1][5][ MOD ][4] | : 15 mod 4 = residual of 15/4 = 3 |

Press [NXT] to get to the next menu set:

| | |
|---|---|
| [3][+/-][ ABS ] | : \|-3\| = 3 |
| [5][+/-][ SIGN ] | : sign(-5) = -1 |
| [2][5][4][0][MANT] | : mantissa of 2540 = 2.540 |
| [2][5][4][0][ XPON] | : exponent of 2540 = 3 |
| [2][.][3][5][ IP ] | : integer part of 2.35 is 2 |
| [2][.][3][5][ FP ] | : fractional part of 2.35 is 0.35 |

Press [NXT] once more to get the final menu set:

| | |
|---|---|
| [1][.][4][5][6][7][ENTER][2][ RND ] | : rounds up 1.4567 to 1.46 |
| [1][.][4][5][6][7][ENTER][2][ TRNC ] | : truncates 1.4567 to 1.45 |
| [2][.][3][FLOOR] | : floor(2.3) = 2 |
| [2][.][3][ CEIL ] | : ceil(2.3) = 3 |
| [4][5][ D→R ] | : $45^o$ = 0.78539 rad |
| [1][.][5][ R→D ] | : 1.5 rad = 85.943669… $^o$. |

Press [NXT] [MTH] to recover the MTH menu.

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**BASE menu**: conversion between various numerical base representations, i.e., hexadecimal, binary, decimal, as well as bit, byte, and logical functions. These functions are mainly of interest to computer scientists. Please consult the calculator documentation for further instructions on how to use these functions. A brief introduction to these operations is presented in Chapter 11.

**PROB menu**: (Press [NXT] to reach this and other menus). Probability and statistic functions, including:

- [COMB] : combinations of y objects taken x at a time (in a combination the order of selection is irrelevant).

- [PERM] : permutations of y objects taken x at a time (in a combination the order of selection is important to consider).

- [ ! ]: for an integer number n, this is the factorial of the number, namely n(n-1)(n-2)…3 2 1 = n! For a real number x, this is the Gamma function, Γ(x-1).

 

- [RAND]: random number generator.  It generates a random number uniformly distributed in [0,1].

- [ RDZ ] : sets the seed for the random number generator.

Press [NXT] to obtain the following menu:

- [UTPC] : Upper-tail probability for the Chi-squared ($\chi^2$) probability distribution.
- [UTPF] : Upper-tail probability for the F probability distribution.
- [UTPN] : Upper-tail probability for the Normal probability distribution.
- [UTPT] : Upper-tail probability for the Student-t probability distribution.
- [NDIST] : Probability density function for the Normal distribution.

Press [MTH] to recover the MTH menu.

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## Combinatorics, random numbers, and probability functions

### Factorials, permutations, and combinations

The *factorial* of an integer *n* is defined as: $n! = n\cdot(n-1)\cdot(n-2)\ldots 3\cdot 2\cdot 1$. By definition, 0! = 1.

Factorials are used in the calculation of the number of permutations and combinations of objects. For example, the number of *permutations* of *r* objects from a set of *n* distinct objects is

$$_{n}P_{r} = n(n-1)(n-1)...(n-r+1) = n!/(n-r)!$$

Also, the number of *combinations* of *n* objects taken *r* at a time is

$$\binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

To simplify notation, I'll use *P(n,r)* for permutations, and *C(n,r)* for combinations.

The HP48 series calculators provide functions to calculate factorials, permutations and combinations by entering [MTH][NXT][*PROB*]. The soft key displays now the functions:

[*COMB*][*PERM*][ *!* ][*RAND*][ *RDZ* ]

The operation of the first three functions is described below:

[*COMB*]: Calculates the number of combinations of n (in level 2) items taken r (in level 1) at a time;
[*PERM*]: Calculates the number of permutations of n (in level 2) items taken r (in level 1) at a time;
[ *!* ]: Factorial of a positive integer (in level 1). For a non-integer, x [ *!* ], returns Γ(x+1), where Γ(x) is known as the *Gamma* function.

For example, to calculate 8!, enter: [8][MTH][NXT][*PROB*][ *!* ].

The display shows a value of 40320.

To calculate P(15,3), enter

[1][5][ENTER][3][ENTER][↰][MTH][NXT][*PROB*][*PERM*],

or, if you are already in the MTH\PROB subdirectory, enter

[1][5][ENTER][3][ENTER][*PERM*].

The result is 2730.

To calculate C(15,3) enter

[1][5][ENTER][3][ENTER][↰][MTH][NXT][*PROB*][*COMB*],

or, if you are already in the MTH\PROB subdirectory, enter

[1][5][ENTER][3][ENTER][*COMB*].

The result is 455.

## The Gamma function

The *Gamma function* is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

This function has the property that ,

$$\Gamma(\alpha) = (\alpha\text{-}1)\ \Gamma(\alpha-1), \text{ for } \alpha > 1,$$

therefore, it can be related to the factorial of a number, i.e.,

$$\Gamma(\alpha) = (\alpha\text{-}1)!,$$

when $\alpha$ is a positive integer.

In the HP 49 G, the factorial function [ ! ] is extended to any real number to define the Gamma function as

$$\Gamma(x) = (x\text{-}1)!$$

Therefore, to obtain $\Gamma(2.5)$, for example, use

[2][.][5][ENTER][1][-][↰][MTH][NXT][PROB][ ! ].

The result is $\Gamma(2.5) = 3.32335097045$.

## Generating random numbers

The HP48G series calculator has a random number generator that returns a real number between 0 and 1. The generator is able to produce sequences of random numbers. However, after a certain number of times (a very large number indeed), the sequence tends to repeat itself. For that reason, the HP48G generator is referred to as a *pseudo-random* number generator. To generate a random number with your calculator, press:

[↰][MTH][NXT][*PROB*][*RAND*].

To generate a sequence of numbers just keep pressing the [*RAND*] soft key.

If you want to generate a sequence of number and be able to repeat the same sequence later, you can change the "seed" of the generator by entering a given number in level 1 and pressing [ *RDZ* ] before generating the sequence. Random number generators operate by starting with a "seed" number that is transformed into the first random number of the series. The current number then serves as the "seed" for the next number and so on. By "re-seeding" the sequence with the same number you can reproduce the same sequence more than once. For example, try the following:

| | |
|---|---|
| [ . ][2][5][ *RDZ* ] | Use 0.25 as the "seed." |
| [*RAND*] | First random number = 0.75285… |
| [*RAND*] | Second random number = 0.51109… |
| [*RAND*] | Third random number = 8.5429…E-2 = 0.085429…. |

Re-start the sequence:

 

[ . ][2][5][ *RDZ* ] Use 0.25 once more as the "seed."
[*RAND*] First random number = 0.75285…
[*RAND*] Second random number = 0.51109…
[*RAND*] Third random number = 8.5429…E-2 = 0.085429…..…

If you press [ *RDZ* ] with no value in the display, the generator will take a number based on the calculator's clock time and use it as the seed.

The pseudo-random number generator provided in your calculator produces random numbers with a *uniform distribution* in the interval [0,1]. To learn more about uniform distributions check out section 5.5 in your textbook.

## Examples of probability calculations for continuous random variables

The probability distribution for a continuous random variable, X, is characterized by a function *f(x)* known as the *probability density function* (pdf). The pdf has the following properties: *f(x) > 0*, for all x, and

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

Probabilities area calculated using the *Cumulative Distribution Function* (cdf), *F(x),* defined by

$$P[X < x] = F(x) = \int_{-\infty}^{x} f(\xi)d\xi.$$

Where *P[X<x]* stands for "the probability that the random variable X is less than the value x".

***Normal distribution pdf***

The expression for the normal distribution pdf is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp[-\frac{(x-\mu)^2}{2\sigma^2}],$$

where $\mu$ is the mean, and $\sigma^2$ the variance of the distribution.

To calculate the value of $f(\mu,\sigma^2,x)$ for the normal distribution, enter the following values: the mean, $\mu$, in level 3; the variance, $\sigma^2$, in level 2; and, the value x in level 1, then enter

[MTH][NXT][PROB][NXT][NDIST].

For example, check that for a normal distribution, f(1.0,0.5,2.0) = 0.20755374. Use the following sequence:

[1][ENTER][ . ][5][ENTER][2][ENTER][ MTH][NXT][*PROB*][NXT][*NDIST*]

***Normal distribution cdf***

The HP48G series calculator has a function UTPN that calculates the upper-tail normal distribution, i.e.,

UTPN(x) = P(X>x) = 1 - P(X<x).

 

To obtain the value of the upper-tail normal distribution UTPN we need to enter the following values: the mean, $\mu$, in level 3; the variance, $\sigma^2$, in level 2; and, the value x in level 1, then enter

[MTH][NXT][*PROB*][NXT][*UTPN*].

For example, check that for a normal distribution, with $\mu = 1.0$, $\sigma^2 = 0.5$, UTPN(0.75) = 0.638163. Use the following sequence:

[1]ENTER][ . ][5][ENTER][ . ][7][5][ENTER][ MTH][NXT][*PROB*][NXT][*UTPN*]

Different probability calculations for normal distributions [X is $N(\mu,\sigma^2)$] can be defined using the function UTPN, as follows:

$P(X<a) = 1 - UTPN(\mu, \sigma^2,a)$

$P(a<X<b) = P(X<b) - P(X<a) = 1 - UTPN(\mu, \sigma^2,b) - (1 - UTPN(\mu, \sigma^2,a))$
$= UTPN(\mu, \sigma^2,a) - UTPN(\mu, \sigma^2,b)$

$P(X>c) = UTPN(\mu, \sigma^2,c)$

Example: Using $\mu = 1.5$, and $\sigma^2 = 0.5$, find (a) P(X<1.0); (b) P(X>2.0); (c) P(1.0<X<2.0).

(a) P(X<1.0) = 1 - P(X>1.0) = 1 - UTPN(1.5, 0.5, 1.0). Enter:

[1][ENTER][1][.][5][SPC][.][5][SPC][1][MTH][NXT][*PROB*][NXT][*UTPN*][-].

The result is P(X<1.0) = 0.239750.

(b) P(X>2.0) = UTPN(1.5, 0.5, 2.0). Enter:

[1][.][5][SPC][.][5][SPC][2] ([MTH][NXT][*PROB*][NXT]) [*UTPN*].

The result is P(X<2.0) = 0.239750.

(c) P(1.0<X<2.0) = F(1.0) - F(2.0) = UTPN(1.5,0.5,1.0) - UTPN(1.5,0.5,2.0). Enter

[1][.][5][SPC][.][5][SPC][1][UTPN] [1][.][5][SPC][.][5][SPC][2][*UTPN*] [-]

The result is P(1.0<X<2.0) = 0.7602499 - 0.2397500 = 0.524998.

***<u>The Student-t distribution</u>***

The Student-t, or simply, the t-, distribution has one parameter $\nu$, known as the degrees of freedom. The probability distribution function (pdf) is given by

$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\cdot\sqrt{\pi\nu}}\cdot(1+\frac{t^2}{\nu})^{-\frac{\nu+1}{2}}, -\infty < t < \infty$$

where $\Gamma(\alpha) = (\alpha-1)!$ is the gamma function defined above.

 

The HP48G/GX provides for values of the upper-tail (cumulative) distribution function for the t-distribution using [UTPT] given the value of t and the parameter ν. The definition of this function is, therefore,

$$UTPT(\nu,t) = \int_t^{\infty} f(t)dt = 1 - \int_{-\infty}^{t} f(t)dt = 1 - P(T \le t)$$

To use this function, enter ν in level 2 and t in level 1, then press [UTPT]. Recall that to get to the probability functions you need to use the keystroke sequence:

[MTH][NXT][*PROB*][NXT].

For example, to calculate UTPT(5, 2.5), use the following:

[5][ENTER][2][.][5][ENTER][*UTPT*]

The result is: UTPT(5,2.5) = 2.7245…E-2
Alternatively, you can use:

[5][SPC][2][.][5][*UTPT*].

Different probability calculations for the t-distribution can be defined using the function UTPT, as follows:

P(T<a) = 1 - UTPT(ν,a)

P(a<T<b) = P(T<b) - P(T<a) = 1 - UTPT(ν,b) - (1 - UTPT(ν,a)) = UTPT(ν,a) - UTPT(ν,b)

P(T>c) = UTPT(ν,c)

***The Chi-squared ($\chi^2$) distribution***

The Chi-squared ($\chi^2$) distribution has one parameter ν, known as the degrees of freedom. The probability distribution function (pdf) is given by

$$f(x)=\frac{1}{2^{\frac{\nu}{2}}\cdot\Gamma(\frac{\nu}{2})}\cdot x^{\frac{\nu}{2}-1}\cdot e^{-\frac{x}{2}},\nu>0,x>0$$

The HP48G/GX provides for values of the upper-tail (cumulative) distribution function for the $\chi^2$-distribution using [UTPC] given the value of t and the parameter ν. The definition of this function is, therefore,

$$UTPC\ (\nu,x)=\int_t^{\infty}f(x)dx=1-\int_{-\infty}^{t}f(x)dx=1-P(X\le x)$$

To use this function, enter ν in level 2 and x in level 1, then press [*UTPC*]. For example, to calculate UTPC(5, 2.5), use the following:

[5][ENTER][2][.][5][ENTER][*UTPC*]

The result is: UTPC(5,2.5) = 0.776495…

Alternatively, you can use:

[5][SPC][2][.][5][*UTPC*].

Different probability calculations for the Chi-squared distribution can be defined using the function UTPC, as follows:

P(X<a) = 1 - UTPC(ν,a)

P(a<X<b) = P(X<b) - P(X<a) = 1 - UTPC(ν,b) - (1 - UTPC(ν,a)) = UTPC(ν,a) - UTPC(ν,b)

P(X>c) = UTPC(ν,c)

***The F distribution***

The F distribution has two parameters νN = numerator degrees of freedom, and νD = denominator degrees of freedom. The probability distribution function (pdf) is given by

$$f(x)=\frac{\Gamma(\frac{\nu N+\nu D}{2})\cdot(\frac{\nu N}{\nu D})^{\frac{\nu N}{2}}\cdot F^{\frac{\nu N}{2}-1}}{\Gamma(\frac{\nu N}{2})\cdot\Gamma(\frac{\nu D}{2})\cdot(1-\frac{\nu N\cdot F}{\nu D})^{(\frac{\nu N+\nu D}{2})}}$$

The HP48G/GX provides for values of the upper-tail (cumulative) distribution function for the F distribution using [UTPF] given the value of F and the parameters νN and νD. The definition of this function is, therefore,

 

$$UTPF(\nu N,\nu D,F) = \int_t^{\infty} f(F)dF = 1 - \int_{-\infty}^{t} f(F)dF = 1 - P(\Im \le F)$$

To use this function, enter νN in level 3, νD in leve2, and F in level 1, then press [UTPF]. For example, to calculate UTPF(10,5, 2.5), use the following:

[1][0][ENTER][5][ENTER][2][.][5][ENTER][UTPF]

The result is: UTPF(5,2.5) = 0.776495…
Alternatively, you can use:

[1][0][SPC][5][SPC][2][.][5][UTPF].

Different probability calculations for the F distribution can be defined using the function UTPF, as follows:

P(F<a) = 1 - UTPF(νN, νD,a)

P(a<F<b) = P(F<b) - P(F<a) = 1 -UTPF(νN, νD,b)- (1 - UTPF(νN, νD,a))
= UTPF(νN, νD,a) - UTPF(νN, νD,b)

P(F>c) = UTPF(νN, νD,a)

---

## FFT menu: Fast Fourier Transform – brief introduction

This menu is useful in transforming a signal (time domain) into the frequency domain, through the function [FFT] (Fast Fourier Transform), or a spectrum in the frequency domain into a signal in the time domain through the function [IFFT] (Inverse Fast Fourier Transform). The signal or spectrum used as input for these functions is provided as a vector with an even number of elements.

Use [MTH][NXT][FFT] to access the FFT menu.

Examples:

[↰][[ ]][1][SPC][2][SPC][3][SPC][4][ENTER][FFT]

Gives the following spectrum:

[(10,0), (-2,2),(-2,0),(-2,-2)]

To see the full result, press [▼], which starts the matrix editor. Use the arrow keys to move from element to element. Press [ENTER] to return to normal calculator display.

With the spectrum in level 1 of the display, press [IFFT], to recover the signal [1 2 3 4].

 

# Calculations with complex numbers

## Introduction to complex numbers

A *complex number z* is a number written as

$$z = x + iy,$$

where *x* and *y* are real numbers, and *i* is the *imaginary unit* defined by $i^2 = -1$.

The complex number *x+iy* has a *real part*,

$$x = Re(z),$$

and an *imaginary part*,

$$y = Im(z).$$

We can think of a complex number as a point P(x,y) in the x-y plane, with the x-axis referred to as the *real axis*, and the y-axis referred to as the *imaginary axis*. Thus, a complex number represented in the form *x+iy* is said to be in its *Cartesian representation*.

A complex number can also be represented in polar coordinates (*polar representation*) as

$$z = re^{i\theta} = r \cdot \cos\theta + i\, r \cdot \sin\theta$$

where

$$r = |z| = (x^2+y^2)^{1/2}$$

is the *magnitude* of the complex number z, and

$$\theta = Arg(z) = \arctan(y/x)$$

is the *argument* of the complex number z.

The relationship between the Cartesian and polar representation of complex numbers is given by the **Euler formula**:

$$re^{i\theta} = \cos\theta + i \sin\theta$$

The *complex conjugate* of a complex number $z = x + iy = re^{i\theta}$,, is

$$\overline{z} = x - iy = re^{-i\theta}.$$

The complex conjugate of z can be thought of as the reflection of z about the real (x-) axis.
Similarly, the *negative* of z,

$$-z = -x-iy = - re^{i\theta},$$

can be thought of as the reflection of z about the origin.

## The CMPLX menus – basic complex number applications

There are two CMPLX (complex numbers) menu available in the calculator.

### CMPLX menu through MTH

The first *CoMPLeX* menu presented herein is accessed by using [↰][MTH][NXT][CMPLX]. The first menu shows the following functions:

[ RE ]: Real part of a complex number
[ IM ]: Imaginary part of a complex number
[ C→R ]: Takes a complex number (x,y) and separates it into its real and imaginary parts.
[ R→C ]: Takes the numbers in registers x and y and forms the complex number (y,x)
[ ABS ]: Calculates the magnitude of a complex number or the absolute value of a real number.
[ ARG ]: Calculates the argument of a complex number.

Press [NXT] to see the second menu:

[ SIGN]: Calculates a complex number of unit magnitude as z/|z|.
[ NEG ]: Changes the sign of the complex number in register x of the display.
[CONJ]: Produces the complex conjugate of the complex number in register x.

### CMPLX menu defined in the keyboard

A second CMPLX menu is accessible by using the right-shift option associated with the [ 1 ] key, i.e., [↱][CMPLX]. The resulting menu include the following functions:

[ ARG ]: Calculates the argument of a complex number.
[ ABS ]: Calculates the magnitude of a complex number or the absolute value of a real number.
[CONJ]: Produces the complex conjugate of the complex number in register x.
[ i ]: Enters the imaginary unit, $i = (-1)^{1/2}$.
[ IM ]: Imaginary part of a complex number
[ NEG ]: Changes the sign of the complex number in register x of the display.

Press [NXT] to see the second menu:

[ RE ]: Real part of a complex number
[ SIGN]: Calculates a complex number of unit magnitude as z/|z|.

## Polar representation of a complex number

To obtain the polar representation of a complex number change the coordinate system to cylindrical. You can perform that change by using: [↰][PRG][MXT][MODES][ANGLE]. This menu will show the current settings for angular measurement and coordinate system highlighted by a rectangle in the corresponding labels. Press [CYLIN] to change to cylindrical (or polar) coordinates.

---

## Examples of basic complex number operations

Make sure your calculator is in RPN mode, angle units set to radians, coordinates set to rectangular, and the CAS is set to Complex. Enter the complex number

$$z = -3+5i,$$

as

[↰][( )][3][+/-][↱][ , ][5][ENTER].

 

Press [ENTER] eight times to place eight copies of the number z in the stack. Then, activate the CMPLX menu from the keyboard, i.e., [↦][CMPLX]. Try the following exercises:

- Find the argument: [ARG], returns $\theta = 2.111\ldots rad$. Use [⇦] to drop this result from the stack.
- Find the magnitude: [ABS], returns $r = 5.83095\ldots$. Use [⇦] to drop this result from the stack.
- Find the conjugate: [CONJ], returns $\bar{z} = (-3.,-5.)$. Use [⇦] to drop this result from the stack.
- Find imaginary part: [ IM ], returns $Im(z) = 5$. Use [⇦] to drop this result from the stack.
- Find the negative of z: [ NEG ], returns $-z = (3., -5.)$.
- Find real part: [NXT][ RE ], returns $Re(z) = -3$. Use [⇦] to drop this result from the stack.
- Find a unit normal vector corresponding to z: [SIGN], returns $u = (-0.514495\ldots, 0.8574\ldots)$. Press [ENTER] to return to normal calculator display.
- To check that the last result is indeed of length 1, use [NXT][ABS]. Use [⇦] to drop result from stack.

**Note**: the functions ABS and ARG can be accessed directly by combining the left- and right-shifts, respectively, with the [ $\div_Z$ ] key.

Activate the alternate CMPLX menu by using [↤][MTH][NXT][CMPLX]

- Separate the complex number into is real and imaginary parts: [ C→R ], resulting in register y = -3., and register x = 5.
- To put the complex number back together use [ R→C ].

***Polar representation***

- To see the polar representation of the number use: [CAT][ALPHA][C], use the down arrow key to highlight CYLIN, then press [OK]. This changes the coordinate system to cylindrical (or polar) coordinates, and results in the number being displayed as $(5.8309, \angle 2.11)$, or $z = 5.83095 \cdot e^{2.111i}$.
- Notice that if the coordinates are set to cylindrical any complex number will automatically be represented in its polar form even if you enter its Cartesian representation. For example, try entering:

  [↤][()][1][↦][ , ][5][ENTER].

  The result returned by the calculator is $(5.09901, \angle 1.373)$.

- To return to the Cartesian representation change the coordinates back to rectangular by using [CAT][ALPHA][R], highlighting RECT, and pressing [OK]. The number will be displayed now as $(1.,5.)$.

## Complex number calculations

Complex numbers can be added, subtracted, multiplied, and divided. The rules for these operations are shown below:

Let

$$z = x + i\cdot y = r\cdot e^{\,i\cdot\theta},$$

$$z_1 = x_1 + i\cdot y_1 = r_1\cdot e^{\,i\cdot\theta_1},$$

and

$$z_2 = x_2 + i\cdot y_2 = r_2\cdot e^{\,i\cdot\theta_2},$$

be complex numbers. In these definitions the numbers $x$, $y$, $x_1$, $x_2$, $y_1$, and $y_2$ are real numbers.

**Addition**: $$z_1 + z_2 = (x_1 + x_2) + i\cdot (y_1 + y_2)$$

**Subtraction**: $$z_1 - z_2 = (x_1 - x_2) + i\cdot (y_1 - y_2)$$

**Multiplication**: $$z_1\cdot z_2 = (x_1\cdot x_2 - y_1\cdot y_2) + i\cdot(x_1\cdot y_2 + x_2\cdot y_1) = r_1\cdot r_2\cdot e^{\,i\cdot(\theta_1 + \theta_2)}.$$

**Multiplication of a number by its conjugate** results in the square of the number's magnitude, i.e.:

$$z\cdot \bar{z} = (x + i\cdot y)\cdot(x - i\cdot y) = x^2 + y^2 = r^2 = |z|^2$$

**Division**:

$$\frac{z_1}{z_2} = \left(\frac{z_1}{z_2}\right)\cdot\left(\frac{\bar{z}_2}{\bar{z}_2}\right) = \frac{z_1\cdot\bar{z}_2}{|z_2|^2} = \frac{x_1\cdot x_2 + y_1\cdot y_2}{x_2^2 + y_2^2} + i\,\frac{y_1\cdot x_2 - x_1\cdot y_2}{x_2^2 + y_2^2} = \frac{r_1}{r_2}\cdot e^{i\cdot(\theta_1-\theta_2)}.$$

**Powers**: $$z^n = (r\cdot e^{\,i\cdot\theta})^n = r^n\cdot e^{\,i\cdot n\theta}$$

**Roots**: because the argument $\theta$ of a complex number $z$ has a periodicity of $2\pi$, we can write

$$z = r\cdot e^{\,i\cdot(\theta+2k\pi)}, \quad \text{for } k = 0, 1, 2, \ldots$$

There are $n$ n-th roots of z calculated as

$$\sqrt[n]{z} = z^{1/n} = r^{1/n}\cdot e^{i\cdot\frac{(\theta+2k\pi)}{n}}, \quad k = 0,1,2,\cdots(n-1).$$

---

***<u>Examples of operations with complex numbers</u>***

To enter complex numbers simply enter them as ordered pairs, as shown earlier. Try the following exercises. Remember to press [ENTER] after entering a complex number:

(-2.,3.) + 5 = (3.,3.)          (4.,2.)+(0,-1.) = (4.,1.)          (-3.2, -5.2) – (6.6, -2.) = (-9.8,-3.2)

(2.,3.)·(-1.,4.) = (-14.,5.)    (2.,3.)/(5.,-2.) = (0.137,0.655)    $(2.,3.)^3$ = (-46., 9.)

√(-1.,5.) = (1.431,1.746)    $(2.,2.)^{-2}$ = (0.,-0.125)

**Note**: roots, other than the square, can not be calculated directly in the HP 49 G. You will have to find the magnitude, *r*, and argument, *θ*, of the number, and then use the expression shown above for $z^{1/n}$. For example, to find the three cubic roots of 1 = (1.,0.). First find *r* = *|(1.,0.)| = 1.0, θ = Arg(1.,0.) = 0.*, thus,

$$\sqrt[3]{z} = z^{1/3} = 1^{1/3} \cdot e^{\frac{2k\pi}{3}\cdot i}, \quad k = 0,1,2.$$

To find the roots use the following:

[EQW][↰][$e^x$][2][×][ALPHA][↰][k][×][↰][ π][×] [↰][ *i* ] [▲][▲] [÷][3][ENTER][ENTER][ENTER]

This will keep three copies of the expression *EXP(2kiπ/3)*. To evaluate the three roots you need to replace values of k = 0, 1, and 2. You can do that replacement by using the evaluation symbol ( | ) available in the [TOOL] key when combined with the right-shift key, i.e., [↱][ | ]. The replacement is defined by using lists of the form {k 0}, {k 1}, and {k 2}.

For k = 0, use:

[↰][{}] [ALPHA][↰][K] [SPC] [0] [ENTER] [↱][ | ] [ENTER] [↱][EVAL]

resulting in the value of 1.0. Use [⇦] to drop this result from the stack.

Try the replacement {k 1}, to get *–(1-i√3)/2*. With [↱][→NUM], the result is *(-0.5,0.8660)*. Use [⇦] to drop this result from the stack.

Finally, try the replacement {k 2}, to get *–(1+i√3)/2*. With [↱][→NUM], the result is *(-0.5,-0.8660)*. Use [⇦] to drop this result from the stack.

---

## Functions of a complex variable

We defined a complex variable *z* as *z = x + iy*, where *x* and *y* are real variables, and $i = (-1)^{1/2}$. We can also define another complex variable

$$w = F(z) = \Phi + i\Psi,$$

where, in general,

$$\Phi = \Phi(x,y), \text{ and } \Psi = \Psi(x,y),$$

are two real functions of *(x, y)*. These real functions can also be given in terms of the polar coordinates (*r,θ)* if we use the polar representation for *z*, i.e.,

$$z = r\cdot e^{i\theta} = r\,(\cos\theta + i\cdot\sin\theta).$$

In such case,

 

$$\Phi = \Phi(r,\theta), \text{ and } \Psi = \Psi(r,\theta).$$

Recall that the coordinate transformations between Cartesian and polar coordinates are:

$$r = (x^2 + y^2)^{1/2}, \qquad \tan\theta = y/x,$$
$$x = r\cos\theta, \qquad y = r\sin\theta$$

The complex variable *w* is also known as a *complex function*. Another name for a complex function is "*mapping*." Thus, we say *F(z)* is a mapping of *z*. In geometric terms, this means that any figure in the x-y plane gets "mapped" onto a different figure on the Φ–Ψ plane by the complex function F(z).

As an example, take the function

$$w = F(z) = \ln(z) = \ln(r\cdot e^{i\theta}) = \ln(r) + i\theta.$$

We can identify the functions

$$\Phi = \Phi(r,\theta) = \ln(r), \text{ and } \Psi = \Psi(r,\theta) = \theta,$$

as the real and imaginary components, respectively, of the function *ln(z).* Using the transformations indicated above we can also write,

$$\Phi = \Phi(x,y) = \ln[(x^2+y^2)^{1/2}] = (1/2)\ln(x^2+y^2), \text{ and } \Psi = \Psi(x,y) = \tan^{-1}(y/x).$$

## Example: Expanding ln(z) using the HP 49 G calculator

The following keystrokes can be used to obtain the real and imaginary parts of the complex function w = ln(z). We will take advantage of this exercise to show you a few of the algebraic functions of the HP 49 G calculator, although we will not discuss such functions in depth until later.
First, type in the function LN(z), then create the equality z = r·EXP(i·θ), as follows:

[↦][ ‘ ][↦][LN][ALPHA][↤][Z][ENTER]
[↦][ ‘ ][ALPHA][↤][Z][↦][=] [ALPHA][↤][R] [×][↤][$e^x$] [↤][ *i* ] [×] [ALPHA][↦][T] [ENTER]

Then, use the SUBSTitute command in the [↦][ALG] menu to replace z with its polar representation:

[↦][ALG][SUBST]

This results in LN(r·EXP(i·θ)). To expand this logarithmic expression, we use the command TEXPA in the [↤][EXP&LN] menu:

[↤][EXP&LN] [TEXPA]

which results in LN(r) + LN(EXP(i·θ)). We know that LN and EXP are inverse functions, but apparently the HP 49 G doesn't, therefore, we help a little by simplifying the result to LN(r) + i·θ. This can be performed by editing the latter result by pressing [▼][EDIT]. Press the right-arrow key [▶] until the cursor is on top of the *i* in the expression. Use the backspace key [⇦] to clear up the characters LN(EXP(. Then, use the right-arrow key, [▶], and the backspace key, [⇦], to clear up the two parentheses left out to the right of *θ* in the expression. Press [ENTER] twice to return to normal calculator display. We have thus separated the function LN(z) into LN(r) + i·θ. (The moral of this story is: *do not expect the calculator to do everything algebraic for you. You need to understand the operation you are performing in the calculator to be able to interpret, and, perhaps, modify, the results given by the machine*.)

 

### Example: Expanding $z^2$ using the HP 49 G calculator

Let's try another example by expanding the function $z^2$, with $z = x + i\cdot y$. Use the equation writer to enter the following expression:

[EQW] [↰][()] [ALPHA][↰][X] [+] [↰][ $i$ ] [×] [↰][ALPHAX][Y] [▶][▶][▶] [$y^x$][2] [ENTER]

This produces the result $(x + i\cdot y)^2$ . To expand the expression use: [↱][EVAL].

The result is: $x^2+2iyx-y^2 = (x^2-y^2)+i(2xy)$.

## Functions of complex numbers in the HP 49 G calculator

Exponential, logarithmic, trigonometric, hyperbolic, and other functions can be applied directly to complex numbers by entering the numbers as ordered pairs, and then entering the appropriate function name. In the following exercises the complex numbers are shown as they will look like when entered in the screen. For example, we would show the value *(3., -2.)*, which is entered as: [↰][( )][3][↱][ , ][2][+/-][ENTER], followed by, say, [↰][$e^x$], to indicate the operation *exp((3.,-2.)) = (-8.35…,-18.26…)*. In traditional Cartesian representation of complex numbers this operation would be written as $exp(3-2i) = -8.35-18.26i$.

Try the following exercises:

(5.,2.) [↰][$e^x$] : $exp(5+2i) = -61.76+134.95i$.

(-1.,2.)[↱][LN]: $ln(-1+2i) = 0.80+2.03i$

(3.,4.) [↰][$x^2$]: $(3+4i)^2 = -7+24i$

(5.,5.) [√x]: $\sqrt{(5+5i)} = 2.46+1.02i$

(-2.,-2) [SIN]: $sin(-2-2i) = -3.42+1.51i$

(0.,-3.) [COS]: $cos(-3i) = 10.07$

(7.5,2.2)[TAN]: $tan(7.5+2.2i) = 0.02+1.02i$

3.5 [ASIN]: $sin\text{-}1(3.5) = 1.57-1.92i$

 

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