A Tutorial of Linear Programming Using True Basic Program OPTLP

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This tutorial aims at explaining the basic calculation procedures for the simplest linear programming problem to the novice in a step by step fashion. Although many computer programs can be used to carried out the similar tasks automatically, it is a good idea for the notice to fully understand the mechanics of the calculations, at least for some typical problems.

1 A Typical Linear Programming Problem

A problem of linear programming (LP) is to find an optimum (minimum or maximum) of an *objective function*, expressed in *linear* form in relation to several *original variables*, subject to one or more linear *constraints*. The constraints can be of equality type, or inequality (greater than or less than) type. The solution of LP problem is due to Geoge Dantzig as published in 1963 as *the Simlex method*. Here we will use a program, OPTLP, by Hanna and Sandall (1995)[1] to solve basic LP problems. Although the calculations are carried out automatically by the computer program, we have to understand what's going on during the calculation process. To this end, we now review several important characteristics of a typical LP problem.

First we write a sample LP problem as

$$\begin{gathered} Minimize \quad z = 4x_1 - 2x_2 - x_3 \\ Subject\ to \\ 3x_1 + 4x_2 - x_3 \leq 1 \\ -2x_1 + 10x_2 + 7x_3 \leq 3 \end{gathered} \tag{1}$$

This sample problem is due to Hanna and Sandall (1995). For typical LP problems, the values of all the *original* variables, namely, x_1, x_2,..., have to be *positive*. If some of them are not positive, new variables have to be introduced to make the resultant *new* LP problem containing *only* positive original variables. Also the RHS (right-hand side) for all the constraints have to be positive. If they are not originally positive, the constraints can be multiplied by -1 and the inequality signs be reversed.

An hypothetical LP problem and its solution strategy are illustrated geometrically in Figure 1. Here, for the LP problem, the constraints form a *convex* set, and the solution of the *linear programming*, if exists, lies only on one of the *corners* of the convex set (case of unique solution), or along a line (case

[1]Hanna O.T. and Sandall O.C., 1995. Computational Methods in Chemical Engineering, Prentice Hall PTR, Upper Saddle River, New Jersey.

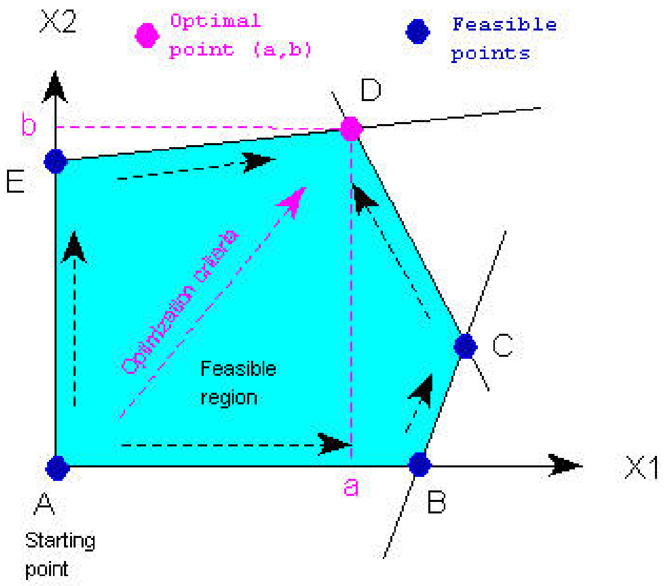

Figure 1: *A Geometric representation of solution to a linear programming problem*

of non-unique solutions)[2]. For the case shown in Figure 1, there are only 2 original variables, x_1 and x_2. As a result, the objective function $z = f(x_1, x_2)$ corresponds to lines in the x_1-x_2 plane, if the value of z is determined. The ultimate aim is to locate a pair of x_1 and x_2, so that an optimal of the objective function is obtained. For convenience, let's only use *mininum* for optimization, because any *maximum* objective function can be transposed into a minimum problem by minimizing the *minus* of the original maximizing objective function.

Let's return to Figure 1 and describe the solution procedure of the LP problems intuitively. Suppose, from some mathematical analysis, we know that the solution of this LP problem lies in one of the 5 corners of the convex set as shown in Figure 1. Then we just need to try the 5 points one by one to see which can lead to the minimization of the objective function. These 5 points are called *feasible* points, meaning that the optimal point has to be chosen from these candidate points. The method is to evaluate the objective function exhaustively at all 5 points and choose the point having the minimum objective function value. This exhaustive searching method is not efficient especially when there are many points to be evaluated. In the simplex method, however, the searching progresses more efficiently toward the final optimal point. Assume first we start with point A. After the objective function is evaluated at this point, a test is used to determine if this is the optimal point. If it is, the search process stops here. If, on the other hand, by evaluating the objective function, there is still possibility to further decrease the value of the objective function, the search goes on. The next point at which to evaluate the objective function is chosen such that the objective function is decreased most. For small problems, the searching process would reach the final optimal point in one or two steps.

Next we will see how the simplex method works by solving the sample LP problem of Eqn(1) step by step. We first assume that the problem only has LT (less than) type of constraints. For problems

[2]For non-linear programming, the solution may be found in the interia of the convex set, which could be tackled either with some other analytical methods, or more practically, with a simple but powerful method called *method of direct search* by Hanna and Sandall in their 1995 book

having other types (equal to or greater than) of constraints, some pre-calculations are required but the fundamental method used is similar to the one we will describe now. In Eqn(1), we need to introduce 2 new variables, here called *slack variables*, x_4 and x_5, so that all the LT constraints now are changed into equalities, as shown in Eqn(2).

$$4x_1 - 2x_2 - x_3 = z \tag{2a}$$
$$3x_1 + 4x_2 - x_3 + x_4 = 1 \tag{2b}$$
$$-2x_1 + 10x_2 + 7x_3 + x_5 = 3 \tag{2c}$$

The variables x_4 and x_5 are called *basic variables* for the now equality constraints to which they belong, while other variables, x_1, x_2 andx_3 are called *non-basic variables.* For this kind LP problem the simplex method first *solve for* the basic variables assuming all the non-basic variables being equal to zero. In this case, we have $x_4 = 1$,and $x_5 = 3$. As now $x_1 = x_2 = x_3 = 0$, the objective function now has the value of zero. So far we have got the first *basic feasible solution* to the original LP problem.

Because the objective function has 2 coefficients in negative (those of x_2 and x_3), it can be expected that further increases in either x_2 or x_3 can decrease the objective function even more. So, this first feasible solution is not an optimal solution. Observe that x_2 has the most negative coefficient, suggesting that increase x_2 would decrease the objective function the most. As a result, we will *exchange* x_2 with one of the current basic variables (e.g., either x_4, or x_5) as a new basic variable. The following criteria determines from which equation this exchange will going to be done: *To exchange x_2 with a current basic variable (in a particular equation) so that a maximum increase of x_2 is permitted by all the constraints.* From Eqn(2), we see the first constraint permit a maximum increase of x_2 to 0.25, while the second constraint allows x_2 to increase to 0.3. So, we choose to exchange x_2 with the current basic variable, x_4. As shown on page 205 of Hanna and Sandall (1995), this can be done by solving x_2 from the first constraint of Eqn(2)(e.g., dividing the first constraint equation by the coefficient of x_2) and substitute into the second constraint and the objective function. This way, x_2 and x_5 will now become the new basic variable, while x_1, x_3, and x_4 are non-basic variables. The simplex procedure can now be repeated once more to obtain a new value for the objective function z. However, it is more rewarding to use the matrix arithmetic to carry out the the steps for the solving-substituting calculations, because this is the way the computer is programmed to do to carry out this kind of tasks. We will show how this works using a method described by Fox et al (1987)[3].

Let's continue with our sample LP problem of Eqn(1). To conduct matrix calculation, we now introduce another variable, x_6 into the objective function and write the objective function following the two equality constraints as follows.

$$3x_1 + 4x_2 - x_3 + x_4 + 0x_5 + 0x_6 = 1 \tag{3a}$$
$$-2x_1 + 10x_2 + 7x_3 + 0x_4 + x_5 + 0x_6 = 3 \tag{3b}$$
$$4x_1 - 2x_2 - x_3 + 0x_4 + 0x_5 + x_6 = 0 \tag{3c}$$

Now there are 3 rather than 2 basic variables in Eqn(3), namely, x_4, x_5, and x_6. From the previous analysis, we know that we want exchange x_2 with one of the current basic variables, x_4. The variable x_6 is totally a "dummy" variable introduced just in order to conduct some matrix calculations, and it is not to be considered as a candidate of a new non-basic variable. it is always a basic variable and we will see later it will hold the value of the objective function z at he end of the simplex calculation. First we using the coefficients of the new basic variables (x_2, x_5, and x_6) to form matrix A_1

$$A_1 = \begin{pmatrix} 4 & 0 & 0 \\ 10 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \tag{4}$$

[3]Fox W.P., Giodano F.R., Maddox S.L., Weir M.D., 1987, Mathematical Modelling with MINITAB. Brooks/Cole Publishing Company, Monterey, California.

and using the coefficients of Eqn (3), including the RHS (but assigning a value 0 for z, because z is undetermined and its value will be stored in x_6), to form A_2.

$$A_2 = \begin{pmatrix} 3 & 4 & -1 & 1 & 0 & 0 & 1 \\ -2 & 10 & 7 & 0 & 1 & 0 & 3 \\ 4 & -2 & -1 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$

Now let we pre-multiply A_2 by the inverse of A_1, e.g., A_1^{-1}. This will complete the variable exchange, with x_2 now becoming a basic variable and x_4 now is a non-basic variable.

$$A_3 = \begin{pmatrix} 0.75 & 1 & -0.25 & 0.25 & 0 & 0 & 0.25 \\ -9.5 & 0 & 9.5 & -2.5 & 1 & 0 & 0.5 \\ 5.5 & 0 & -1.5 & 0.5 & 0 & 1 & 0.5 \end{pmatrix} \tag{6}$$

With this updated coefficient matrix, the original Eqn (2) now is changed into

$$5.5x_1 - 1.5x_3 + 0.5x_4 - 0.5 = z \tag{7a}$$
$$0.75x_1 + x_2 - 0.25x_3 + 0.25x_4 = 0.25 \tag{7b}$$
$$-9.5x_1 + 9.5x_3 - 2.5x_4 + x_5 = 0.5 \tag{7c}$$

Note the objective function z now has the value of -0.5 (rather then 0), indicating the change of variable indeed improved the objective function. This process of exchanging the variables and updating the coefficient matrix and objective function will be repeated until there is no more possibility to improve the value of the objective function. At this point we have reached the optimal solution.

2 An Example for Animal Rationing

Now let's solve a practical LP problem for animal rationing using the program OPTLP by Hanna and Sandall (1995). The problem is the Exercise 2 in Chapter 8 of Fox et al. (1984):

A rancher has determined that the minimum weekly nutritional requirement for an average size horse include 40 lb of protein, 20 lb of carbohydrates, and 45 lb of roughage. These are obtained from the following sources in varying amounts at the given prices, as shown in Table 1. Formulate a mathematical model to determine how to meet the minimum nutritional requirement at minimum cost.

Solution. Assume the farmer is going to use x_1 bales of hay, x_2 sacks of oats, x_3 blocks of feeding blocks and x_4 sacks of high-protein concentrate. We formulate the model as a LP problem,

$$Minimize \quad z = 1.8x_1 + 3.5x_2 + 0.4x_3 + x_4$$

$$\textit{Subject to}$$

$$0.5x_1 + x_2 + 2x_3 + 6x_4 \geq 40 \tag{8}$$

$$2x_1 + 4x_2 + 0.5x_3 + x_4 \geq 20$$

$$5x_1 + 2x_2 + x_3 + 2.5x_4 \geq 45$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Run program OPTLP and we find: $z_{opt} = 17$, $x_1 = 5$, $x_2 = 0$, $x_3 = 20$, $x_4 = 0$. So the optimal cost of the weekly house feedstuff is 17 dollars, with 5 bales of hay and 20 feeding blocks but without using any of oats or high-protein concentrate. Formulation of this problem assumes we have had enough

Table 1: Nutrition requirements of an average size horse.

$Feed\,stuff$	$Protein(lb)$	$Carbohydrates(lb)$	$Roughage(lb)$	$Cost(dollars)$
$Hay\,(per\,bale)$	0.5	2.0	5.0	1.80
$Oats\,(per\,sack)$	1.0	4.0	2.0	3.50
$Feeding\,blocks\,(per\,block)$	2.0	0.5	1.0	0.40
$High-protein\,concentrate\,(per\,sack)$	6.0	1.0	2.5	1.00
$Requirements\,per\,horse\,(per\,week)$	40.0	20.0	45.0	

previous knowledge about animal nutrient requirements. This is usually available, at least approximately for experienced managers. If, however, the price of one or more feedstuff changes, we can re-run the model with the changed price(s) as the new input, and get the updated optimal solution to the problem. Other possibilities can also be tried using this model. For example, if there is a limited supply of feeding blocks and the maximum amount available to this farm per week is 10 blocks, then we have to add an additional *supply constraint* of feeding blocks to the problem formulation, and re-run the model to get an optimized solution. We can imagine that, if other aspects of the model keep unchanged, then the new optimal solution might have to include the more expensive oats or high-protein concentrate in order to satisfy the minimum weekly requirement of horse, which might increase the weekly cost (current optimal cost is 17 dollars) somewhat.

This computer program of linear programming may be used as a "calculator" to tackle other, probably a little larger, problems in agriculture.

3 The Input/Output of OPTLP

Now we add a couple of lines on the use of this program. The users are required to provide data to the subroutine *Init*. These are:

- N, number of original variables.
- M, total number of inequality/equation constraints.
- GT, number of 'greater than' inequality constraints.
- EQ, number of 'equal to' constraints.
- LT, number of 'less than' inequality constraints.
- $A(i, j)$, coefficients of the constraints. These only include the left-hand side coefficients, row by row. First enter all the 'greater than' coefficients, followed by 'equal to' coefficients, and last the 'less than' coefficients.
- $b(i)$, column vector of the right-hand side of the coefficients, in the same order from GT, to EQ, and then to LT.
- $c(j)$, row vector of the coefficients of the objective function.

For the final output, if an unique solution is found (sometimes there can be no solution, either due to inconsistencies in the constraints, or other aspects of the problem formulation), we can see the basic variables selected for the final iteration are printed along with the final value for z. The non-basic variables (including the ones that were originally the basic variables at the start of the problem) have the value of zero.